On the Equivariant Tamagawa Number Conjecture

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Abstract

For a finite Galois extension $K/\mathbb{Q}$ of number fields with Galois group G and a motive $M = M' \otimes h^0(\mathrm{Spec}(K))(0)$ with coefficients in $\mathbb{Q}[G]$, the equivariant Tamagawa number conjecture relates the special value $L^*(M,0)$ of the motivic L-function to an element of $K_0(\mathbb{Z}[G];\mathbb{R})$ constucted via complexes associated to M. The conjecture for nonabelian groups G is very much unexplored. In this thesis, we will develop some techniques to verify the conjecture for Artin motives and motives attached to elliptic curves. In particular, we consider motives $h^0(\mathrm{Spec}(K))(0)$ for an A_4-extension $K/\mathbb{Q}$ and, $h^1(E \times \mathrm{Spec}(L))(1)$ for an S_3-extension $L/\mathbb{Q}$ and an elliptic curve $E/\mathbb{Q}$.

Contents

Chapter 1

Introduction

Special values of L-functions attached to objects arising in number theory and arithmetic geometry are known (or expected) to carry a rich source of information. There has been an enormous amount of research during the past 150 years dedicated to the study of these special values, resulting in beautiful theorems and remarkable conjectures. Notable examples are the analytic class number formula and the conjecture of Birch and Swinnerton-Dyer. In this thesis, we present two approaches to verify a conjecture of Burns and Flach concerning the special values of L-functions attached to motives with coefficients.

Let k be a number field and let X be a smooth projective variety over k. For integers $n, r \in \mathbb{Z}$ with $n \geq 0$ let M be the (pure) Chow motive $h^n(X)(r)$, which comes equipped with deRham, Betti and l-adic realizations. Let A be a finitely generated semisimple $\mathbb{Q}$-algebra that acts on M. One can then define an equivariant motivic L-function $L(M, s)$ attached to M that encompasses the action of A on the motive (see [8] or [23]). The equivariant Tamagawa number conjecture ([8] Conj. 4) relates the leading coefficient in the Taylor expansion of $L(M, s)$ at $s = 0$ to an algebraic element arising from perfect complexes attached to M. The conjecture on the one hand generalizes conjectures of Stark, Chinburg, Gross, Rubin and others, and on the other hand generalizes the conjecture of Birch and Swinnerton-Dyer.

Our interest is in two particular motives $h^0(\mathrm{Spec}(K))(0)$ and $h^1(E_K)(1)$ where K/k is a finite Galois extension and E_K is the base change of an elliptic curve defined over k. The formulation of the equivariant conjecture itself depends on various other conjectures, many of which are known in these two special cases that we are interested in. We shall briefly explain below the key aspects and the importance of these cases.

1.1 The motive $h^0(\mathrm{Spec}(K))(0)$

In the first setting, we have a finite Galois extension K/k of number fields with Galois group $\mathrm{Gal}(K/k) = G$. Let S be a finite set of primes in K stable under the action of G. For an irreducible complex character χ of G, let $L_S(\chi, s)$ denote the Artin L-function and let $L_S^*(\chi, 0)$ denote the leading coefficient in the Taylor series expansion of $L_S(\chi, s)$ at $s = 0$. Then one can view $L_S(s) := (L_S(\chi, s))_{\chi \in \widehat{G}}$ as a function with values in $\prod_{\chi \in \widehat{G}} \mathbb{C} = \zeta(\mathbb{C}[G])$, where ζ is the center and $\widehat{G}$ is the set of irreducible complex characters of G. Further, $L_S^*(0) = (L_S^*(\chi, 0))_{\chi \in \widehat{G}}$ lies in $\zeta(\mathbb{R}[G])^\times$.

There exists a natural map arising from a long exact sequence of K-theory

$$\widehat{\delta} : \zeta(\mathbb{R}[G])^\times \to K_0(\mathbb{Z}[G], \mathbb{R}),$$

where $K_0(\mathbb{Z}[G], \mathbb{R})$ is the relative K_0 group (cf. [8]). Using Tate sequences one can construct an element $R\Omega(K/k) \in K_0(\mathbb{Z}[G], \mathbb{R})$. We let $T\Omega(K/k) = R\Omega(K/k) - \widehat{\delta}(L_S^*(0))$. The equivariant Tamagawa number conjecture for the motive $M = h^0(\mathrm{Spec}(K))_k$ (with an action by the semisimple algebra $A = \mathbb{Q}[G]$) states that $T\Omega(K/k) = 0$. The vanishing of $T\Omega(K/k)$ in $K_0(\mathfrak{M}, \mathbb{R})$, where $\mathfrak{M}$ is a maximal $\mathbb{Z}$-order containing $\mathbb{Z}[G]$, is equivalent to the strong Stark conjecture as stated in [14]. Also, the vanishing of $T\Omega(K/k)$ in $K_0(\mathbb{Z}[G])$ is equivalent to the central conjectures of [12]. The equivariant conjecture also recovers several refinements of the Stark conjecture due to Chinburg, Gross, Rubin, and others (cf. [6]).

For nonabelian extensions K/k, the vanishing of $T\Omega(K/k)$ is known for certain dihedral extensions (cf. [5]) and for an infinite family of quaternion extensions (cf. [9]). One of the key ideas used in [9] is that the map

$$K_0(\mathbb{Z}[G], \mathbb{R}) \to K_0(\mathfrak{M}, \mathbb{R}) \times K_0(\mathbb{Z}[G^{\mathrm{ab}}], \mathbb{R}) \times K_0(\mathbb{Z}[G])$$

is injective for $G = Q_8$, the quaternion group (here $\mathfrak{M}$ is a maximal $\mathbb{Z}$-order containing $\mathbb{Z}[G]$). However, this fails to hold for $G = A_4$ (see Appendix A).

In this thesis, we shall restrict to a particular extension $K/\mathbb{Q}$ with $\mathrm{Gal}(K/\mathbb{Q}) \simeq A_4$, and develop some techniques for computing $R\Omega(K/\mathbb{Q})$. We use the ideas of Chinburg ([12]) to construct the Tate sequence and explicitly compute the determinants arising from this sequence to evaluate $R\Omega(K/\mathbb{Q})$, and verify the vanishing of $T\Omega(K/\mathbb{Q})$. One could possibly extend these techniques to verify the conjecture for an infinite family of A_4-extensions.

1.2 The motive $h^1(E_K)(1)$

In this case we consider an elliptic curve E defined over k and look at the base change curve $E_K := E \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K)$. There is a natural action of G on E_K, and we let $A = \mathbb{Q}[G]$. The associated motivic L-function is $L(E_K, s) = (L(E \otimes \chi, s))_{\chi \in \widehat{G}}$, which takes values in $\prod_{\chi \in \widehat{G}} \mathbb{C} \simeq \zeta(\mathbb{C}[G])$. Because of the center of symmetry, the most interesting special value of this L-function is

$$L^*(E_K, 1) = (L^*(E \otimes \chi, 1))_{\chi \in \widehat{G}} \in \zeta(\mathbb{R}[G])^\times,$$

where we write L^* to denote the leading coefficient in the Taylor expansion of the L-funciton at $s = 1$. Part of the equivariant conjecture claims that $\widehat{\delta}(L^*(E_K, 1))$, which *a priori* is an element of $K_0(\mathbb{Z}[G]; \mathbb{R})$, is in fact an element of $K_0(\mathbb{Z}[G]; \mathbb{Q})$. This fact is known in several cases due to Shimura and others (cf. [33], [4]).

Now, for a rational prime l let $T_l := \varprojlim_n E(\bar{\mathbb{Q}})/l^n$ and let $V_l := T_l \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$. Let S_l be a finite set of primes containing the infinite primes, primes over l and primes of bad reduction. One can then define a perfect $\mathbb{Z}_l[G]$-complex $R\Gamma_c(\mathcal{O}_{K,S_l}, T_l)$. Via the height pairing, period isomorphism and comparison isomorphism we can associate to this complex an element

$$R\Omega(E, \mathbb{Z}_l[G]) \in K_0(\mathbb{Z}_l[G]; \mathbb{Q}_l).$$

The equivariant conjecture claims that the l-part of $\widehat{\delta}(L^*(E_K, 1))$ equals $R\Omega(E, \mathbb{Z}_l[G])$. This conjecture generalizes the reputed conjecture of Birch and Swinnerton-Dyer (see [22] for details).

In this thesis, we give a naive approach to verify the conjecture for the motive $h^1(E_K)(1)$. We use modular symbols to compute special values of L-functions with abelian twists, and use analytic methods to get numerical values of the same, with nonabelian twists. Under some hypotheses we will use these L-values to give numerical evidence to the conjecture.

The thesis is organized as follows. In Chapter 2 we briefly define the terms involved and state the equivariant conjecture. In Chapter 3 we reinterpret the conjecture for Artin motives, and give a detailed description of a method to verify the conjecture in the nonabelian setting. Much of this Chapter is part of [27]. Further, we apply this method and prove the conjecture in a particular case. In Chapter 4 we consider the motives arising from the base change of elliptic curves. We give the details of computing L-values using modular symbols, and using analytic methods. We apply these to give numerical evidences to the conjecture (again in a nonabelian setting). This Chapter is part of the article [28]. In the appendix, we give a brief account of the history of various results and conjectures related to the special values of L-functions that eventually led to the formulation of the equivariant conjecture.

1.3 Notations

- For a group G, we denote by $\widehat{G}$ the set of all irreducible complex characters of G.
- For a number field k, we denote by G_k the Galois group of $\bar{k}/k$.
- For a finite place l in a number field k, we denote by Fr_l the Frobenius element attached to l in G_k.
- S_∞ = set of all infinite places (of the underlying number field).
- For a number field k, we denote by $\mathcal{O}_k$ the ring of integers in k. Further, for a finite set S of primes, $\mathcal{O}_{k,S}$ denotes the ring of S-integers.
- For any ring R, we let $\zeta(R)$ denote the center of R.
- For an R-module M, we let M^* to be the R-dual $Hom_R(M,R)$.
- Given a ring homomorphism $\phi : R \to S$ and an R-module M, we let $M_S := M \otimes_R S$.
- For a ring R, we let R^{op} denote the opposite ring.

Chapter 2

The Conjecture

2.1 Some algebra

2.1.1 Algebraic K-groups

For a ring R (with unit element), we let $\mathrm{PMod}(R)$ denote the category of finitely generated projective R-modules. We define $K_0(R)$ to be the abelian group generated by the symbols $[P]$ for each $P \in \mathrm{PMod}(R)$, with relations $[P_1]+[P_3]=[P_2]$ for every short exact sequence

$$0 \to P_1 \to P_2 \to P_3 \to 0.$$

The group $K_1(R)$ is defined to be the abelianization of the direct limit $\varinjlim_n GL_n(R)$. Equivalently, one can define $K_1(R)$ to be the abelian group generated by symbols $[P,\alpha]$ for every $P \in \mathrm{PMod}(R)$ and $\alpha \in \mathrm{Aut}_R(P)$, with relations

$$[P_1,\alpha_1]+[P_2,\alpha_2]=[P_3,\alpha_3],$$

for every commutative diagram of short exact sequences

$$0 \to (P_1,\alpha_1) \to (P_2,\alpha_2) \to (P_3,\alpha_3) \to 0,$$

(cf. [17]).

Note that a ring homomorphism $\phi : R \to S$ induces homomorphims $\phi_i^* : K_i(R) \to K_i(S)$, for $i = 0,1$. One can define a relative K-group $K_0(R,\phi)$, and morphisms $K_1(S) \to K_0(R,\phi)$

and $K_0(R,\phi) \to K_0(R)$ such that

$$K_1(R) \to K_1(S) \to K_0(R,\phi) \to K_0(R) \to K_0(S)$$

is exact. In terms of generators and relations, the group $K_0(R,\phi)$ is generated by triples $[M,N;\lambda]$, where $M,N \in \mathrm{PMod}(R)$ and $\lambda : M \otimes_R S \to N \otimes_R S$ is an isomorphism of S-modules, and with relations given by short exact sequences (see [17] §40B for details). When the morphism ϕ is evident we write $K_0(R;S)$ for the group $K_0(R,\phi)$. Also, for an R-module M, we let $M_S := M \otimes_R S$.

Now, let F be a field and let R be a central simple algebra over F. Fix an extension F'/F such that $R' := R \otimes_F F' \simeq M_n(F')$ and an indecomposable idempotent $e \in R'$. Let V be a finitely generated R-module and let $\phi \in \mathrm{End}_R(V)$. We define the reduced rank $\mathrm{rr}_R(V)$ of V as

$$\mathrm{rr}_R(V) := \dim_{F'}(e(V \otimes_F F'))$$

and the reduced determinant $\mathrm{detred}_R(\phi)$ of ϕ as

$$\mathrm{detred}_R(\phi) = \det_{F'}(\phi \otimes 1 | e(V \otimes_F F')).$$

The reduced rank induces a morphism

$$\mathrm{rr}_R : K_0(R) \to \mathbb{Z}.$$

The $\mathrm{detred}_R(\phi)$ is an element $F^\times$, which is independent of the choices of F' and e. It induces a morphism, called the reduced norm map,

$$\mathrm{nr}_R : K_1(R) \to F^\times$$

(cf. [17]).

2.1.2 Virtual objects

Let R be any arbitrary ring. As before, let $\mathrm{PMod}(R)$ denote the category of finitely generated projective R-modules. In [18], Deligne has constructed a category $V(R)$ of virtual objects and a universal determinant functor

$$[\ ] : \mathrm{PMod}(R) \to V(R)$$

satisfying certain conditions as in [8]. The above functor naturally extends to a functor

$$[\ ] : D^p(R) \to V(R)$$

where $D^p(R)$ is the category of perfect complexes of R-module. A complex $C^\bullet$ of R-module is called *perfect* if there is a chain

$$C^\bullet \to C_1^\bullet \leftarrow C_2^\bullet \cdots \to P^\bullet$$

of quasi-isomorphisms, where $P^\bullet$ is a bounded complex of finitely generated projective modules.

It follows from the proof of the existance of $V(R)$ in [18] that there are isomorphisms

$$K_i(R) \xrightarrow{\sim} \pi_i(V(R))$$

for $i = 0, 1$ where $K_i(R)$ denotes the algebraic K-group associated to R (see [31] for details), and $\pi_0(V(R))$ is the group of isomorphism classes of objects of $V(R)$ and $\pi_1(V(R)) = \mathrm{Aut}_{V(R)}(1_{V(R)})$.

Given a finitely generated subring S of $\mathbb{Q}$, an S-order $\mathfrak{A}$ of a finite dimensional $\mathbb{Q}$-algebra A is a finitely generated S-module such that $\mathfrak{A} \otimes_{\mathbb{Z}} \mathbb{Q} = A$. For any such S-order, and any field extension F of $\mathbb{Q}$, one has a notion of relative virtual objects $V(\mathfrak{A}; F)$ and $V(\mathfrak{A}_p; \mathbb{Q}_p)$, where $\mathfrak{A}_p = \mathfrak{A} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ (see [8] for details). There are isomorphisms

$$\pi_0(V(\mathfrak{A}; F)) \xrightarrow{\sim} K_0(\mathfrak{A}; F)$$

and

$$\pi_0(V(\mathfrak{A}_p; \mathbb{Q}_p)) \xrightarrow{\sim} K_0(\mathfrak{A}_p; \mathbb{Q}_p),$$

which are compatible with the Mayor-Vietoris sequences (cf. Prop. 2.5 in [8]).

Remarks.

1. Let R be a commutative semisimple ring. Then an R-module P is projective if and only if it is locally free at all the prime ideals of R. In this case, the determinant functor can be defined locally by

$$\mathrm{Det}_{R_{\mathfrak{p}}}(P_{\mathfrak{p}}) = \left(\wedge^{\mathrm{rank}_{R_{\mathfrak{p}}}(P_{\mathfrak{p}})} P_{\mathfrak{p}}, \mathrm{rank}_{R_{\mathfrak{p}}}(P_{\mathfrak{p}})\right)$$

   for every prime $\mathfrak{p} \in \mathrm{Spec}(R)$. Note that thus defined $\mathrm{Det}_R(P)$ is a graded line bundle. Let $\mathcal{P}(R)$ denote the category of graded line bundles over R. In the above notation, if $\mathfrak{A}$ is a finite flat commutative $\mathbb{Z}$-algebra then there is a natural equivalence of categories $\mathcal{P}(\mathfrak{A}) \xrightarrow{\sim} V(\mathfrak{A})$.

2. Even if R is not commutative, but is semisimple, one can construct the determinant functor in a similar fashion by looking at the indecomposable idempotents. We give this construction below since it is key to some of our computations.

   By Wedderburn's decomposition we can assume that R is a central simple algebra over a field F. So $R \simeq M_n(D)$ for some division ring D with center F. Further, by fixing an exact Morita equivalence $\mathrm{PMod}(M_n(D)) \to \mathrm{PMod}(D)$, we may assume that $R = D$. Fix a field extension F'/F such that $D \otimes_F F' \simeq M_d(F')$. Let e be an indecomposable idempotent of $M_d(F')$ and let $e_1, \ldots, e_d$ be an ordered F'-basis of $eM_d(F')$. Let V be a finitely generated projective (and hence free) D-module. Let $\{v_1, \ldots, v_r\}$ be a D-basis of V. Set $b := \wedge e_i v_j$. This is an F'-basis of $\mathrm{Det}_{F'}(e(V \otimes_F F'))$. Since any change in the basis $\{v_i\}_i$ multiplies b by an element of $F^\times$, the F-space spanned by b yields a well-defined graded F-line bundle. This defines the determinant functor.

2.2 L-functions

Let X be a smooth projective variety defined over a number field k. For integers n, r with $n \geq 0$, let M be the pure (Chow) motive $h^n(X)(r)$. The realizations of M are

- $H_{dR}(M) := H^n_{dR}(X/k)$, a filtered k-space with its natural decreasing filteration $\{F^i H^n_{dR}(X/k)\}_{i \in \mathbb{Z}}$ shifted by r;

- $H_l(M) := H^n_{et}(X \times_k \bar{k}, \mathbb{Q}_l(r))$, a compatible system of l-adic representations of G_k;

- $H_\sigma(M) := H^n(\sigma X(\mathbb{C}), (2\pi i)^r\mathbb{Q})$, for each $\sigma \in \mathrm{Hom}(k, \mathbb{C})$ a $\mathbb{Q}$-Hodge structure over $\mathbb{R}$ or $\mathbb{C}$ according to whether $v(\sigma)$ is real or complex.

Let A be a finitely generated semisimple $\mathbb{Q}$-algebra that acts on M. It therefore acts naturally on all the realizations of M. We shall denote the motive by M_A to indicate the action of A. For a finite place v of k of residue characteristic p, and a prime $l \neq p$ define

$$L_v(M_A, T) := \mathrm{detred}_{A_l}(1 - \mathrm{Fr}_v^{-1} | H_l(M_A)^{I_v}) \in \zeta(A_l)[T].$$

A compatibility conjecture of Tate implies that this element belongs to $\zeta(A)[T]$ and is independent of the choice of l. We shall henceforth assume this. We can now define

$$L(M_A, s) = \prod_v L_v(M_A, Nv^{-s}).$$

For any finite set of primes S (containing the infinite primes), set

$$L_S(M_A, s) = \prod_{v \notin S} L_v(M_A, Nv^{-s}).$$

The above products converge in the half plane $\mathrm{Re}(s) >> 0$. Our interest is to relate the special values of these L-functions to algebraic properties of M_A.

Examples.

1. Let K/k be finite Galois extension of number fields, and let G be its Galois group. Then, there is a natural action of $A = \mathbb{Q}[G]$ on the motive $M = h^0(\mathrm{Spec}(K))(j)$. The relative L-function $L_S(M_A, s)$ attached to this motive is the tuple $(L_S(\chi, j+s))_{\chi \in \widehat{G}}$ of Artin L-functions attached to the irreducible complex characters of G. This tuple takes values in
$$\prod_{\chi \in \widehat{G}} \mathbb{C} \simeq \zeta(\mathbb{C}[G]) \simeq \zeta(A \otimes_{\mathbb{Q}} \mathbb{C}).$$

2. Let E be an elliptic curve defined over k, and let K/k be a finite Galois extension of number fields with Galois group G. The group G and hence the algebra $A = \mathbb{Q}[G]$ acts on the motive $M = h^1(E \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K))(1)$. The L-function $L(M_A, s)$ attached to this motive is the tuple $(L(E \otimes \chi, 1+s))_{\chi \in \widehat{G}}$ of Hasse-Weil L-functions twisted by irreducible complex characters of G.

2.3 The isomorphism ϑ_∞

For $v \in S_\infty$, the Deligne cohomology is the cohomology of the complex

$$H_v(M) \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\alpha_v} (H_v(M) \otimes_{\mathbb{Q}} \bar{k}_v)^{G_v}/F^0$$

induced by the inclusion $H_v(M) \otimes_{\mathbb{Q}} \mathbb{R} \hookrightarrow H_v(M) \otimes_{\mathbb{Q}} \bar{k}_v$. The comparison isomorphism

$$H_v(M) \otimes_{\mathbb{Q}} \bar{k}_v \xrightarrow{\sim} H_{dR}(M) \otimes_{k,v} \bar{k}_v$$

is G_v-equivariant and hence the above complex can be written as

$$\oplus_{v \in S_\infty} (H_v(M) \otimes_{\mathbb{Q}} \mathbb{R})^{G_v} \xrightarrow{\alpha_M} \oplus_{v \in S_\infty} H_{dR}(M) \otimes_{k,v} k_v/F^0.$$

For the motive $M = h^n(X)(r)$ one can define *motivic cohomology* $H^i(k, M)$ and its finite parts $H^i_f(k, M)$ (see [8] for details).

Conjecture 1. *There exists an exact sequence of finite-dimensional $A_{\mathbb{R}}$-spaces*

$$0 \to H^0(k, M) \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\epsilon} \ker \alpha_M \xrightarrow{r_B^*} H^1_f(k, M^*(1)) \otimes_{\mathbb{Q}} \mathbb{R})^*$$

$$\xrightarrow{\delta} H^1_f(k, M) \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{r_B} \operatorname{coker} \alpha_M \xrightarrow{\epsilon^*} (H^0(k, M^*(1)) \otimes_{\mathbb{Q}} \mathbb{R})^* \to 0$$

where ϵ is the cycle class map into the singular cohomology, r_B is the Beilinson regulator map, and δ is the height pairing.

We set

$$\begin{aligned} \Xi(M) \quad := \quad & [H^0_f(k, M)] \boxtimes [H^1_f(k, M)]^{-1} \boxtimes [H^1_f(k, M^*(1))^*] \\ & \boxtimes [H^0_f(k, M^*(1))^*]^{-1} \boxtimes \boxtimes_{v \in S_\infty} [H_v(M)^{G_v}]^{-1} \boxtimes [H_{dR}(M)/F^0] \end{aligned}$$

where $[\ ] : D^p(A) \to V(A)$ is the determinant functor as defined in the previous section.

The above conjectured exact sequences gives an isomorphism

$$\vartheta_\infty : \Xi(M) \otimes_A A_{\mathbb{R}} \simeq 1_{V_{A_{\mathbb{R}}}}.$$

Remark. Since $H^0_f(M)$ is defined as the image of the cycle class map it is known to be finite dimensional. The vector space $H^1_f(M)$ is also expected to be finite dimensional, but this is not known in general. It is known in the cases we are interested in (and in some trivial cases). For $M = h^0(\mathrm{Spec}(K))(r)$, one has $H^1_f(M) \simeq K_{2r-1}(\mathcal{O}_K) \otimes_{\mathbb{Z}} \mathbb{Q}$, which is known to be finite dimensional by a Theorem of Quillen (cf. [31]). Further, for a smooth projective curve X/k and $M = h^1(X)(1)$, one has $H^1_f(M) = \mathrm{Pic}^0(X)(k) \otimes_{\mathbb{Z}} \mathbb{Q}$, which is known to be finite dimensional by Mordell-Weil Theorem.

Examples.

1. For $M = h^0(\mathrm{Spec}(K))(0)$, K/k a finite Galois extension of number fields with Galois group G, one has

   $$\Xi(M) = [U_{S_\infty} \otimes_{\mathbb{Z}} \mathbb{Q}] \boxtimes [X_{S_\infty} \otimes_{\mathbb{Z}} \mathbb{Q}]^{-1}$$

   where for a finite set S of primes, U_S denotes the group of S-units in K, and $X_S = \ker(\oplus_{v\in S}\mathbb{Z} \to \mathbb{Z})$. The map α_M is the regulator map

   $$U_{S_\infty} \otimes_{\mathbb{Z}} \mathbb{R} \to X_{S_\infty} \otimes_{\mathbb{Z}} \mathbb{R},$$

   which is defined by

   $$u \mapsto -\sum_{v\in S} \log |u|_v \cdot v.$$

   The isomorphism $\vartheta_\infty : \Xi(M) \otimes \mathbb{R}[G] \simeq 1_{V_{\mathbb{R}[G]}}$ is induced by the map α_M.

2. For $M = h^1(E_K)(1)$, where E_K is the base change of an elliptic curve E defined over k, one has

   $$\Xi(M) = [E(K) \otimes_{\mathbb{Z}} \mathbb{Q}] \boxtimes [E(K)^* \otimes_{\mathbb{Z}} \mathbb{Q}] \boxtimes [H^0(E_K, \Omega^1)^*] \boxtimes [H_1(E_K(\mathbb{C}), \mathbb{Q})^+]$$

   where $*$ denotes the dual and $+$ denotes the submodule fixed by the complex conjugation. In this case, the isomorphism ϑ_∞ is obtained via the (Néron-Tate) height pairing and the period isomorphism.

2.4 The isomorphism ϑ_l

Let $\mathfrak{A}$ be an R-module in A. An $\mathfrak{A}$-submodule T of V is said to be an $\mathfrak{A}$-lattice in V if it is both finitely generated and full (that is, $T \otimes_{\mathfrak{A}} A = V$).

An $\mathfrak{A}$-structure T on M is a set $\{T_v : v \in S_\infty\}$ where, for each $v \in S_\infty, T_v$ is an $\mathfrak{A}$-lattice in $H_v(M)$ and for each prime $l \in \mathrm{Spec}(R)$ the image T_l of $T_v \otimes_{\mathbb{Z}} \mathbb{Z}_l$ under the comparison isomorphism $H_v(M) \otimes_{\mathbb{Q}} \mathbb{Q}_l \simeq H_l(M)$ is both independent of v and G_k-stable. An $\mathfrak{A}$-structure is projective if each T_v is a projective $\mathfrak{A}$-module.

Let T be a projective $\mathfrak{A}$-structure on M. For any prime l let $S_l = S \cup \{\text{primes above } 1\}$. For any continuous G_S-module N let

$$\begin{aligned} R\Gamma(\mathcal{O}_{k,S_l}, N) &:= C^\bullet(G_{S_l}, N), \\ R\Gamma_c(\mathcal{O}_{k,S_l}, N) &:= \mathrm{Cone}\left(R\Gamma(\mathcal{O}_{k,S_l}, N) \to \oplus_{v \in S_l} C^\bullet(G_v, N)\right)[-1]. \end{aligned}$$

One has a conjectural isomorphism

$$\vartheta_l : A_l \otimes_A \Xi(M) \xrightarrow{\sim} [R\Gamma_c(\mathcal{O}_{k,S_l}, V_l)] \simeq A_l \otimes_{\mathfrak{A}_l} [R\Gamma_c(\mathcal{O}_{k,S_l}, T_l)].$$

We refer the reader to [8] Section 3 for the details about the complex $R\Gamma_c$. The isomorphism ϑ_l defines an element $([R\Gamma_c(\mathcal{O}_{K,S_l}, T_l)], \Xi(M); \vartheta_l)$ in $V(\mathfrak{A}_l) \times_{V(A_l)} V(A)$. The isomorphism class of this element is independent of the choice of the set S or the choice of the projective structure T, and therefore we can define under certain coherence hypothesis (cf. [8])

$$(\Xi(M), \vartheta_\infty) := \left(\prod_p [R\Gamma_c(\mathcal{O}_{K,S_l}, T_l)], \Xi(M), \prod_p \vartheta_p; \vartheta_\infty \right).$$

We shall denote the isomorphism class of this element in $\pi_0(V(\mathfrak{A}; \mathbb{R})) \simeq K_0(\mathfrak{A}; \mathbb{R})$ by $R\Omega(M, \mathfrak{A})$.

Examples.

1. As before, our first example is $M = h^0(\mathrm{Spec}(K))(0)$, where K/k is a finite Galois extension of number fields. In this case $U_S \subset U_S \otimes \mathbb{Q}$ and $X_S \subset X_S \otimes \mathbb{Q}$ are Galois stable $\mathbb{Z}[G]$-modules, but they are not projective. However, the Tate sequences associated to the extension K/k provides a canonical projective $\mathbb{Z}[G]$-structure using which one can formulate the conjecture. We will see this in detail in the next chapter.

2. Our second example is $M = h^1(E_K)(1)$, where E_K is base change of an elliptic curve defined over k. In this case $H(\sigma E_K(\mathbb{C}), \mathbb{Z}) \subset H(\sigma E_K(\mathbb{C}), \mathbb{Q})$ defines a projective $\mathbb{Z}[G]$-structure. In Chapter 4, we will consider the conjecture for this particular projective $\mathbb{Z}[G]$-structure.

2.5 Equivariant conjecture

Consider the long exact sequence in K-theory

$$\begin{tikzcd} \cdots \arrow[r] & K_1(A_{\mathbb{R}}) \arrow[r, "\delta"] \arrow[d, "\mathrm{nr}"] & K_0(\mathfrak{A}; \mathbb{R}) \arrow[r] & K_0(\mathfrak{A}) \arrow[r] & \cdots \\ & \zeta(A_{\mathbb{R}})^{\times} & & & \end{tikzcd}$$

In [8], Burns and Flach have constructed a canonical map

$$\widehat{\delta} : \zeta(A_{\mathbb{R}})^{\times} \quad \longrightarrow \quad K_0(\mathfrak{A}; \mathbb{R}) \tag{2.1}$$

such that $\widehat{\delta} \circ \mathrm{nr} = \delta$. As we saw in the Section 2.2, the leading coefficient of the L-function attached to the motive M lies in $\zeta(A_{\mathbb{R}})^{\times}$. The equivariant conjecture relates the image of this leading coefficient under the map $\widehat{\delta}$ to the object $R\Omega(M, \mathfrak{A}) \in K_0(\mathfrak{A}; \mathbb{R})$ constructed in the previous section.

Conjecture 2 (Burns-Flach [8]). *With the setup as above one has the following:*

1. *The motivic L-function $L(M, s)$ can be analytically continued to $s = 0$.*

2. *Regarding* $\mathrm{ord}_{s=0} L(M, s)$ *as a locally constant function on* $\mathrm{Spec}(\zeta(A_{\mathbb{C}}))$ *one has*

$$\mathrm{ord}_{s=0} L(M, s) = \mathrm{rr}_A(H^1_f(K, M^*(1))^*) - \mathrm{rr}_A(H^0_f(K, M^*(1))^*)$$

   where rr_A *is the reduced rank map defined in Section 2.1.*

3. *(Rationality) Set*

$$\begin{aligned} L^*(M, 0) &:= lim_{s \to 0} s^{-\mathrm{ord}_{s=0} L(M,s)} L(M, s) \\ L(M, \mathfrak{A}) &:= \widehat{\delta}(L^*(M, 0)) \in K_0(\mathfrak{A}; \mathbb{R}) \end{aligned}$$

and

$$T\Omega(M,\mathfrak{A}) := L(M,\mathfrak{A}) + R\Omega(M,\mathfrak{A}).$$

Then, $T\Omega(M,\mathfrak{A}) \in K_0(\mathfrak{A};\mathbb{Q})$.

4. *(Integrality)* $T\Omega(M,\mathfrak{A}) = 0$.

Remarks.

1. The above conjecture is a generalization to the motives with noncommutative coefficients of conjecture formulated by Fontaine and Perrin-Riou, and Kato (cf. [23],[25]).

2. The conjecture of Bloch and Kato is the special case of this conjecture with $\mathfrak{A} = \mathbb{Z}$.

In the next two chapters we shall consider two particular motives and give a reinterpretation of the conjecture.

Chapter 3

Artin Motives

3.1 Reinterpretation of the conjecture

3.1.1 The conjecture

Let K/k be a finite Galois extension of number fields with Galois group G. Our interest is in the motive $M = h^0(\mathrm{Spec}(K))(0)$, which we regard as a motive defined over k with coefficients in $A = \mathbb{Q}[G]$.

As noted in the Chapter 2, the L-function associated with this motive is a tuple of Artin L-functions. To be precise, let S_k be a finite set of primes in k, and let $S := S_K$ be the set of primes in K lying above the primes in S_k. If $\rho : G \to GL(V)$ is a complex representation corresponding to a character $\chi \in \widehat{G}$ then

$$L_S(\chi, s) := \prod_{\mathfrak{p} \in S_k} \det(1 - \mathrm{Fr}_{\mathfrak{P}} N\mathfrak{p}^{-s} | V^{I_{\mathfrak{P}}})^{-1}$$

is the Artin L-function (relative to S) attached to the character χ. Here $\mathfrak{P}$ is an arbitrary prime in K lying above $\mathfrak{p}$ and $I_{\mathfrak{P}}$ is the inertia group of $\mathfrak{P}$. The motivic L-function is then given by

$$L_S(M, s) = (L_S(\chi, s))_{\chi \in \widehat{G}}.$$

This L-function has a meromorphic continuation to the whole s-plane and therefore we can consider its Taylor series expansion at $s = 0$. For the rest of this chapter we shall supress the motive M from our notation and write $L_S^*(0)$ for the leading coefficient

$$L_S^*(M, 0) = (L_S^*(\chi, 0))_{\chi \in \widehat{G}} \in \prod_{\chi \in \widehat{G}} \mathbb{R} \simeq \zeta(\mathbb{R}[G])^{\times}.$$

Now, on the arithmetic side we have

$$\Xi(M) = [H_f^0(k, M)] \boxtimes [H_f^1(k, M^*(1))^*] \boxtimes \boxtimes_{v \in S_\infty} [H_v(M)^{G_v}]^{-1}.$$

The other terms that appear in the definition of $\Xi(M)$ are trivial. One has

$$H_f^0(k, M) := H^0(k, M) \simeq \mathbb{Q},$$

and

$$\oplus_{v \in S_\infty} H_v(M)^{G_v} \simeq (\oplus_{v \in S_\infty} \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

For an subring $\mathfrak{A}$ of $\mathbb{Q}[G]$, let $\mathfrak{A}^\#$ denote the image of $\mathfrak{A}$ under the $\mathbb{Q}$-linear map

$$\begin{array}{rcl} \# : \mathbb{Q}[G] & \longrightarrow & \mathbb{Q}[G] \\ \sum_{g \in G} a_g g & \mapsto & \left(\sum_{g \in G} a_g g \right)^\# := \sum_{g \in G} a_g g^{-1}. \end{array}$$

For a $\mathfrak{A}^{op}$-module P, let $P^\# = \mathfrak{A}^\# \otimes_{\mathfrak{A}^{op}} P$. Then, one has

$$H_f^1(k, M^*(1))^* \simeq (U_{S_\infty} \otimes_{\mathbb{Z}} \mathbb{Q})^{*\#}.$$

For any finite set T of primes in K, let X_T denote the kernel of the augmentation map

$$\begin{array}{rcl} \oplus_{v \in T} \mathbb{Z} & \longrightarrow & \mathbb{Z} \\ (a_v)_{v \in T} & \mapsto & \sum_{v \in T} a_v. \end{array}$$

Then, from the above observation, we get

$$\Xi(M) = [(U_{S_\infty} \otimes_{\mathbb{Z}} \mathbb{Q})^{*\#}] \boxtimes [(X_{S_\infty} \otimes_{\mathbb{Z}} \mathbb{Q})^{*\#}]^{-1}.$$

To relate the special values of the L-function relative to S, we consider the following relative version of $\Xi(M)$:

$$\Xi_S(M) = [(U_S \otimes_{\mathbb{Z}} \mathbb{Q})^{*\#}] \boxtimes [(X_S \otimes_{\mathbb{Z}} \mathbb{Q})^{*\#}]^{-1}.$$

Now, to consider the integrality part of the equivariant conjecture, we want to find projective $\mathbb{Z}_l[G]$-structures that give the necessary algebraic element in $K_0(\mathbb{Z}_l[G];\mathbb{Q}_l)$. The following Theorem of Tate allows us to replace U_S and X_S by $\mathbb{Z}[G]$-modules of finite projective dimension.

Theorem 1 (Tate [35]). *Let S be large enough so that it contains all the infinite primes, ramified primes and so that the S-class number Cl_S is coprime to the order $|G|$ of the Galois group G. Then there exists an exact sequence*

$$0 \to U_S \to A \to B \to X_S \to 0 \tag{3.1}$$

of $\mathbb{Z}[G]$-modules representing a canonical class in $\mathrm{Ext}^2_G(X_S, U_S)$, with A and B being of finite projective dimension over $\mathbb{Z}[G]$.

By the above Theorem, we get an element

$$[A, B; \psi_{S,\mathbb{R}}] \in K_0(\mathbb{Z}[G];\mathbb{R})$$

where $\psi_{S,\mathbb{R}}$ is obtained by the scalar extension of the exact sequence (3.1) and the regulator map

$$R_S : U_S \otimes_{\mathbb{Z}} \mathbb{R} \to X_S \otimes_{\mathbb{Z}} \mathbb{R}.$$

The integrality part of Conjecture 2 for the motive $M = h^0(\mathrm{Spec}(K))(0)$ with coefficients in $A = \mathbb{Q}[G]$ can now be restated as

Conjecture 3. $\widehat{\delta}(L_S^*(0)^{\#}) - [A, B; \psi_{S,\mathbb{R}}] = 0$ *in* $K_0(\mathbb{Z}[G];\mathbb{R})$.

Remark. By Lemma 5 in [8], we have that the equivariant conjecture does not depend on the choice of S. Therefore it is enough to prove the conjecture for a particular set S of primes.

3.1.2 Known cases

Theorem 2 (Burns, Flach, Greither [10, 22]). *For a finite abelian extension $K/\mathbb{Q}$ one has $T\Omega(K/\mathbb{Q}) = 0$.*

So, the interest naturally is to verify the conjecture in nonabelian cases. In [5], an algorithm is developed to verify the conjecture in the dihedral case. Using this algorithm, it is shown that the conjecture is true for all Galois extensions $L/\mathbb{Q}$ with $\mathrm{Gal}(L/\mathbb{Q})$ the dihedral group of order 6. The arguments involve reducing the conjecture to an abelian case using the map

$$K_0(\mathbb{Z}[D_n],\mathbb{R})_{tors} \to K_0(\mathbb{Z}[C_n],\mathbb{R})_{tors},$$

which is proven to be injective.

So, the next group to be considered is the Quaternion group Q_8. In this case one has

Lemma 1 (Burns, Flach [9]). *The natural map*

$$K_0(\mathbb{Z}[Q_8],\mathbb{R}) \to K_0(\mathfrak{M}_{Q_8},\mathbb{R}) \times K_0(\mathbb{Z}[Q_8]) \times K_0(\mathbb{Z}[Q_8^{\mathrm{ab}}],\mathbb{R}),$$

where $\mathfrak{M}_{Q_8}$ is a maximal order containing $\mathbb{Z}[Q_8]$, is injective.

Thus the vanishing problem in $K_0(\mathbb{Z}[Q_8],\mathbb{R})$ reduces to a vanishing problem in the three groups on the right hand side. The vanishing in $K_0(\mathfrak{M}_{Q_8},\mathbb{R})$ is Strong Stark, and is known due to Tate, since all the characters of Q_8 are rational. The vanishing in $K_0(\mathbb{Z}[Q_8])$ is Chinburg's Ω-conjecture. This is known for an infinite family of extensions constructed by Chinburg in [12]. In [9], the vanishing in $K_0(\mathbb{Z}[Q_8^{\mathrm{ab}}],\mathbb{R})$ is shown for this infinite family of extensions. Thus, the conjecture is known for an infinite family of Q_8-extensions.

We are therefore interested in the next nonabelian case, that is, the conjecture over A_4-extensions.

The above Lemma fails to hold for A_4 (see Appendix for details). Therefore, the question arises whether we can directly compute $T\Omega(K/\mathbb{Q})$ by constructing the Tate Sequence. The answer is yes and this is what we shall achieve by the end of this Chapter.

Results similar to Theorem 2 for abelian extensions over imaginary quadratic fields are known due to Bley and Johnson (see [3] and [24]).

3.2 Preliminaries

3.2.1 Chinburg's idea

Let $K/\mathbb{Q}$ be a finite Galois extension with Galois group G. For any place v of K denote by G_v the decomposition group of v. Suppose that S is a finite set of primes in K satisfying the following properties:

1. Cl_S is coprime to $\#G$.

2. S is stable under the G action.

3. S contains all the infinite primes and all the primes in K which ramify.

4. There exists $v_0 \in S$ for which $G_{v_0} = G$.

Let S_0 be a set of representatives for the G-orbits in S. For $v \in S_0$ let

$$0 \to \underline{\mathrm{Y}}_v(-2) \to \underline{\mathrm{A}}_v \to \underline{\mathrm{B}}_v \to \mathbb{Z} \to 0$$

be an exact sequence of G_v-modules in which the middle terms are free and finitely generated. Inducing the above sequence from G_v to G, we get

$$0 \to Y_v(-2) \to A_v \to B_v \to Y_v \to 0,$$

an exact sequence of G-modules. The existence of v_0 implies that we can identify X_S with $\oplus_{v \in S_0 \setminus \{v_0\}} Y_v$. Thus summing all the sequences above over the set $S_0 \setminus \{v_0\}$ we get

$$0 \to X(-2) \to \oplus_{v \in S_0 \setminus \{v_0\}} A_v \to \oplus_{v \in S_0 \setminus \{v_0\}} B_v \to X_S \to 0,$$

where $X(-2) = \oplus_{v \in S_0 \setminus \{v_0\}} Y_v(-2)$.

For $v \in S_0$ let $S(v)$ be a set of representatives for G_v-orbits of S such that $S_0 \subset S(v)$. Let J_S denote the group of S-ideles and U_S denote the group of S-units in K. Then by class field theory (cf. [1], [32]) we have

$$\mathrm{Ext}^2_G(Y_v, J_S) = H^2(G_v, J_S) = \oplus_{w \in S(v)} H^2(G_v \cap G_w, K_w^*)$$

where K_w is the localization of K at w. For a subgroup H of G let $\mathrm{inv}(H, w) : H^2(H, K_w^*) \to \mathbb{Q}/\mathbb{Z}$ be the invariant map. The image of this map is generated by $\frac{1}{\#H}$.

The inclusion $U_S \to J_S$ induces an injection (cf. [1], p. 64)

$$\mathrm{Ext}^2_G(Y_v, U_S) = H^2(G_v, U_S) \to H^2(G_v, J_S).$$

The image consists of all $\beta \in H^2(G_v, J_S) = \oplus_{w \in S(v)} H^2(G_v \cap G_w, K_w^*)$ satisfying

$$\sum_{w \in S(v)} \mathrm{inv}(G_v \cap G_w, w)(\beta) = 0.$$

Now, for each $v \in S_0 \setminus \{v_0\}$, choose a map $f_v : Y_v(-2) \to U_S$ corresponding to $\beta \in H^2(G_v, J_S) = \oplus_{w \in S(v)} H^2(G_v \cap G_w, K_w^*)$ such that

$$\mathrm{inv}(G_v \cap G_w, w)(\beta) = \begin{cases} \frac{1}{\#(G_v \cap G_w)} & \text{if } w = v, \\ -\frac{1}{\#(G_v \cap G_w)} & \text{if } w = v_0, \\ 0 & \text{otherwise.} \end{cases}$$

Combining all such f_v's we get a map $f : X(-2) \to U_S$. This map f represents an extension class $\beta \in \mathrm{Ext}^2_G(X_S, U_S)$. Then by [12], Prop. 3.2.1, β corresponds to the same extension class as that of a Tate sequence:

$$0 \to U_S \to A \to B \to X_S \to 0.$$

Now, let N be a free $\mathbb{Z}[G]$-module such that there exists a map $\tilde{f} : X(-2) \oplus N \to U_S$, which is surjective. Then we have the following exact sequences:

$$0 \to \ker(\tilde{f}) \to X(-2) \oplus N \to U_S \to 0,$$

$$0 \to X(-2) \to A_X \to B_X \to X_S \to 0,$$

where $A_X = \oplus_{v \in S_0 \setminus \{v_0\}} A_v$ and $B_X = \oplus_{v \in S_0 \setminus \{v_0\}} B_v$. Then following Tate's argument (cf. [35], Thm. 5.1) we get a Tate sequence:

$$0 \to U_S \to (A_X \oplus N)/\ker(\tilde{f}) \to B_X \to X_S \to 0.$$

3.2.2 The cohomology group $H^2(V_4, K_v^*)$

In this subsection we shall assume that $K/\mathbb{Q}$ is a Galois extension with Galois group $G \simeq A_4$. Let V_4 be the Klein four group sitting inside G. We shall compute the elements of the group $H^2(V_4, K_v^*)$ as morphisms arising from some V_4-resolution of $\mathbb{Z}$. By local class field theory (cf. [32]) we know that $H^2(V_4, K_v^*)$ is cyclic of order $\#V_4 = 4$. We will be interested in knowing a sufficient condition on morphisms so that they correspond to the trivial element and a generator of $H^2(V_4, K_v^*)$.

Let $V_4 = \{1, g_1, g_2, g_3\}$. Consider the exact sequence of $\mathbb{Z}[V_4]$-modules:

$$0 \to M \xrightarrow{\delta_3} \mathbb{Z}[V_4] \oplus \mathbb{Z}[V_4] \xrightarrow{\delta_2} \mathbb{Z}[V_4] \xrightarrow{\delta_1} \mathbb{Z} \to 0,$$

defined by

- $\delta_1(1) = 1$
- $\delta_2((1,0)) = 1 - g_2, \delta_2((0,1)) = 1 - g_3$
- $M = \ker(\delta_2)$ and δ_3 is the inclusion

Now, the $\ker(\delta_2)$ is generated by

$$\beta_1 = (-1 - g_1, 1 + g_1), \beta_2 = (1 + g_2, 0), \beta_3 = (0, 1 + g_3).$$

Thus $M \simeq \frac{\mathbb{Z}[G_1] \oplus \mathbb{Z}[G_2] \oplus \mathbb{Z}[G_3]}{N}$ where $G_i \simeq V_4/\{1, g_i\}$. We denote by α_i the generator (i.e, the multiplicative identity) of $\mathbb{Z}[G_i]$. Then N is generated by $(1 + g_1)(\alpha_3) - (1 + g_3)(\alpha_2 + \alpha_1)$.

Let K_v/E_w be an extension of local fields with the Galois group V_4. Let $F_{w_i}, i = 1, 2, 3$ be the fixed fields corresponding to the subgroups $\{1, g_i\}$. We also assume that F_{w_i}/E_w is ramified for $i = 2, 3$ and unramified for $i = 1$. Let N_i denote the norm map $N_{F_{w_i}/E_w}$.

Now, let $\theta : M \to K_v^*$ be a morphism of V_4-modules. Let $x_i = \theta(\alpha_i)$. Then by the V_4-action on M we get the following relations among x_i's:

- $x_i \in F_{w_i}^*$,
- $N_3(x_3) = N_2(x_2) N_1(x_1)$.

Now, θ defines a class in $H^2(V_4, K_v^*)$. This class of θ is trivial if there exists $y_2, y_3 \in K_v^*$ such that $x_i = N_{K_v/F_{w_i}}(y_i)$ where $y_1 = y_2^{-1} y_3$.

Consider the extension K_v/F_{w_1}. We know that $F_{w_1}^*/N(K_v^*)$ is cyclic of order 2. Choose $\beta_1 \in F_{w_1}^* \setminus N(K_v^*)$. Let $\beta = \frac{\beta_1}{g_3(\beta_1)}$. Thus, $N_1(\beta) = 1$.

Lemma 2. *Define $\theta_1 : M \to K_v^*$ by $\theta_1(\alpha_2) = \theta_1(\alpha_3) = 1$ and $\theta_1(\alpha_1) = \beta$. Then the class of θ_1 in $H^2(V_4, K_v^*)$ is the element of order two.*

Proof. First of all note that θ_1 is indeed a morphism between V_4-modules. Also, $(2\theta_1)(\alpha_1) = \beta^2 = N_{K_v/F_{w_1}}(\beta)$ and $N_{K_v/F_{w_3}}(\beta) = 1$. So, by choosing $y_2 = 1$ and $y_3 = \beta$, we see that $2\theta_1$ defines the trivial class in $H^2(V_4, K_v^*)$. Thus, to prove the lemma it is enough to show that the class of θ_1 is not trivial.

Suppose the contrary. Then there exists $y_2, y_3 \in K_v^*$ such that $N_{K_v/F_{w_2}}(y_2) = N_{K_v/F_{w_3}}(y_3) = 1$ and $N_{K_v/F_{w_1}}(y_2^{-1}y_3) = \beta$. Thus, we can find $\eta_2, \eta_3 \in K_v^*$ such that $y_i = \frac{\eta_i}{g_i(\eta_i)}, i = 1, 2$. Then,

$$\beta = (y_2^{-1}y_3) \cdot g_1(y_2^{-1}y_3) = \frac{g_2(\eta_2)\eta_3}{\eta_2 g_3(\eta_3)} \cdot \frac{g_3(\eta_2)g_1(\eta_3)}{g_1(\eta_1)g_2(\eta_3)} = \frac{\beta_1'}{g_3(\beta_1')},$$

where $\beta_1' = \eta_3 g_1(\eta_3)g_2(\eta_2)g_3(\eta_2) \in N_{K_v/F_{w_1}}(K_v^*)$. Then $\beta_1 = \delta\beta_1'$ for some $\delta \in E_w$.

But, F_{w_1}/E_w is an unramified extension and therefore δ is a square modulo a uniformizer of F_{w_1}. By this, it follows that it is a norm of an element from K_v. This implies that $\delta\beta_1' = \beta_1$ is a norm of an element from K_v a contradiction. This completes the proof of the Lemma. ☐

Now, let π be a uniformizer of E_w, π_i be uniformizers of F_{w_i} respectively. Since K_v/F_{w_i} is unramified for $i = 2, 3$ it follows that $\pi_i \notin N_{K_v/F_{w_i}}(K_v^*), i = 2, 3$. Further, $n = N_2(\pi_2^{-1})N_3(\pi_3)$ is a unit in E_w. Since F_{w_1}/E_w is unramified we can find $x \in F_{w_1}$ such that $N_1(x) = n$.

Lemma 3. *Define $\theta_0 : M \to K_v^*$ by $\theta_0(\alpha_1) = x, \theta_0(\alpha_i) = \pi_i$, for $i = 2, 3$. Then the class defined by θ_0 in $H^2(V_4, K_v^*)$ is a generator.*

Proof. It is clear that θ_0 is a V_4-module morphism. By the construction, it is also evident that the class of θ_0 is neither trivial nor equal to that of θ_1. This completes the proof of the Lemma. ☐

3.2.3 The local class group $Cl(\mathbb{Z}[A_4])$

Let N be a (global) field, R be a Dedekind domain with fractional field N, A be a finite dimensional N-algebra and Λ be an R-order in A.

Definition. The *class group* $Cl(\Lambda)$ is defined by the generators

$$\{[X] | X \text{ is a f.g. proj. } \Lambda\text{-module such that } X_p \simeq \Lambda_p \text{ for all } p \in \mathrm{Spec}(\Lambda)\}$$

and relations

$$[X] = [Y] \Leftrightarrow X \oplus \Lambda \simeq Y \oplus \Lambda \text{ (or equivalently, } [X] = [Y] \text{ in } K_0(\Lambda)).$$

The addition is defined by $[X] + [Y] = [Z] \Leftrightarrow X \oplus Y \simeq Z \oplus \Lambda$.

If Λ' is a maximal R-order containing Λ then there exists a natural surjection of the class groups

$$Cl(\Lambda) \to Cl(\Lambda').$$

Definition. The kernel of the above map is called the *kernel group* of Λ and is denoted by $D(\Lambda)$.

Remarks.

1. For $A = N[G]$ and $\Lambda = R[G], G$ a finite group, $Cl(\Lambda) \simeq \ker(K_0(\Lambda) \to K_0(A))$.

2. If $N = \mathbb{Q}, R = \mathbb{Z}, A$ is an algebraic extension of $\mathbb{Q}$ and $\Lambda = \mathcal{O}_A$, then $Cl(\Lambda)$ is the ideal class group of $\mathcal{O}_A$.

3. The kernel group is independent of the choice of Λ'.

4. One has $Cl(M_n(\Lambda)) \simeq Cl(\Lambda)$ and $D(M_n(\Lambda)) \simeq D(\Lambda)$.

Theorem 3 (Endo-Hironaka [21]). *The kernel group $D(\mathbb{Z}[A_4])$ is trivial.*

Proposition 1. *The class group $Cl(\mathbb{Z}[A_4])$ is trivial.*

Proof. Let $\mathfrak{M}$ be a maximal $\mathbb{Z}$-order in $\mathbb{Q}[A_4]$ containing $\mathbb{Z}[A_4]$. Then by the above Theorem $Cl(\mathbb{Z}[A_4]) \simeq Cl(\mathfrak{M})$. Since $\mathbb{Q}[A_4] \simeq \mathbb{Q} \times \mathbb{Q}(\zeta_3) \times M_3(\mathbb{Q})$ one can take $\mathfrak{M} = \mathbb{Z} \times \mathbb{Z}[\zeta_3] \times M_3(\mathbb{Z})$ and therefore $Cl(\mathfrak{M}) \simeq Cl(\mathbb{Z}) \times Cl(\mathbb{Z}[\zeta_3]) \times Cl(M_3(\mathbb{Z})) = 0$ by the above remarks. This completes the proof of Proposition 1. $\square$

3.3 Construction of the Tate sequence

In this section we shall construct the extension $K/\mathbb{Q}$ for which we shall verify the conjecture in question, and construct a Tate sequence using the machinery developed in the previous section. We will be using this Tate sequence to prove the vanishing of $T\Omega(K/\mathbb{Q})$ for this particular extension.

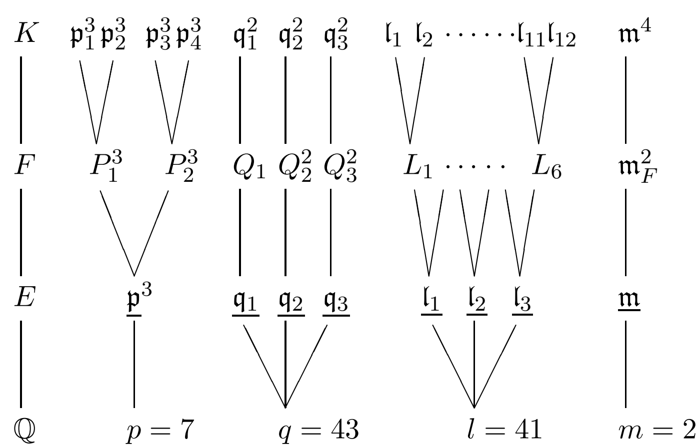
3.3.1 A tetrahedral extension $K/\mathbb{Q}$

Let E be the unique degree 3 subextension of $\mathbb{Q}[\zeta_7]$. Thus, E is generated by a root of the polynomial $X^3+X^2-2X-1=0$. Let $\epsilon_7=\zeta_7+\zeta_7^{-1}$. Then $E=\mathbb{Q}[\epsilon_7]$. It is known that the class group of E is trivial. Now, since, $43 \equiv 1 \pmod 7$ the prime 43 splits in E. Let $\theta_1=2\epsilon_7^2-\epsilon_7-5$. Then, θ_1 generates a prime ideal above 43 in E. Let θ_2 and θ_3 be the conjugates of θ_1. Set $F=F_1=E[v_1]$, where $v_1^2=\theta_2/\theta_3>0$. Let K be the normal closure of F. Then $G(K/\mathbb{Q})=A_4$.

First let us fix some notations. Let F_2 and F_3 denote the field extensions of E in K that are isomorphic to F_1. In other words, these two fields are obtained by adjoining v_2 and v_3 respectively, where $v_2^2=\theta_3/\theta_1$ and $v_3^2=\theta_1/\theta_2$.

Let $G=G(K/\mathbb{Q})=A_4$. Let g_1,g_2 and g_3 be the order two elements of G such that the fixed field corresponding to $\{1,g_i\}$ is F_i. Let $h\in G$ be an element of order three such that h maps F_1 to F_3.

Let $S_{\mathbb{Q}}=\{2,7,41,43,\infty\}$ and for any extension $N/\mathbb{Q}$ let $S_N=\{\text{primes over } S_{\mathbb{Q}} \text{ in } N\}$. Let $S=S_K$. Set $p=7, q=43, l=41, m=2$. The decomposition of these primes in K and its subfields is as follows:



So, the set of primes which ramify in K is $\{2,7,43\}$. The class group of K is of order 16 and the cyclic factors are of orders 4, 2 and 2. The set S defined above satisfies all the conditions stated at the beginning of the previous section. In fact, the prime $l=41$ is chosen so that the S-class group Cl_S is trivial. The unique prime $\mathfrak{m}$ in K above 2 satisfies $G_{\mathfrak{m}}=G$.

We shall now construct the map $f: X(-2)\to U_S$ using which we shall explicitly get the Tate sequence. Recall that $S_{\mathbb{Q}}=\{2,7,41,43,\infty\}$ and that $S=\{\text{primes over } S_{\mathbb{Q}}\}$. Let

$S_0 = \{\mathfrak{m}, \mathfrak{p} = \mathfrak{p}_1, \mathfrak{q} = \mathfrak{q}_1, \mathfrak{l} = \mathfrak{l}_1, \infty\}$ be a set of representatives for G-orbits.

As we have seen in the previous section, the map f is the sum of f_v's for $v = \mathfrak{p}, \mathfrak{q}, \mathfrak{l}, \infty$. One can induce the maps f_v from $\underline{\mathrm{f}}_v : \underline{\mathrm{Y}}_v(-2) \to U_S$. This map is trivial for $v = \mathfrak{l}$ and thus we only need to consider the primes $\mathfrak{p}, \mathfrak{q}$ and ∞.

3.3.2 Case $v = \mathfrak{q}$

First of all, we shall fix a set of representatives for G_v-orbits. Let $S(v) = \{\mathfrak{m}, \mathfrak{p}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3\} \cup \{\text{three primes above } l\} \cup \{\text{three infinite primes}\}$. From the previous section, we have $\underline{\mathrm{Y}}_v = M$. Thus, we need to construct a map $\underline{\mathrm{f}}_v : M \to U_S$ such that the corresponding extension class $\beta \in \mathrm{Ext}^2_G(Y_v, U_S)$ satisfies

$$\mathrm{inv}(G_v \cap G_w, w)(\beta) = \begin{cases} \frac{1}{\#(G_v \cap G_w)} & \text{if } w = v, \\ -\frac{1}{\#(G_v \cap G_w)} & \text{if } w = \mathfrak{m}, \\ 0 & \text{if } w \in S(v) \setminus \{v, \mathfrak{m}\}. \end{cases}$$

We have

$$G_v \cap G_w = \begin{cases} V_4 & \text{if } w = \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{m}, \\ \{1, g_i\} & \text{for some } i, \text{ if } w | \infty, \\ \{1\} & \text{otherwise.} \end{cases}$$

Recall that $\sum_{w \in S(v)} \mathrm{inv}(G_v \cap G_w, w)(\beta) = 0$. Therefore, if $\mathrm{inv}(V_4, v)(\beta) = 1/4$ and $\mathrm{inv}(V_4, w)(\beta) = 0$ for $w = \mathfrak{q}_2, \mathfrak{q}_3$ and for the infinite primes, then it follows that $\mathrm{inv}(V_4, \mathfrak{m}) = -1/4$. In other words we need to construct $\underline{\mathrm{f}}_v$ such that $\mathrm{inv}(V_4, v)(\beta)$ generates $H^2(V_4, K_v^*)$, $\mathrm{inv}(V_4, w)(\beta)$ is trivial in $H^2(V_4, K_w^*)$ for $w = \mathfrak{q}_2, \mathfrak{q}_3$ and such that $\mathrm{inv}(C_2, w)(\beta)$ is trivial in $H^2(C_2, \mathbb{C}^*)$ for the infinite primes w in $S(v)$.

Note that M is generated by α_1, α_2 and α_3 as indicated in the previous section. Thus, the construction of $\underline{\mathrm{f}}_v$ requires finding three S-units u_1, u_2 and u_3 satisfying the following properties (here Q_{ij} denotes the prime ideal in F_i lying below $\mathfrak{q}_j$):

1. $u_i \in F_i$ for $i = 1, 2, 3$.

2. $N_{F_1/E}(u_1) N_{F_2/E}(u_2) = N_{F_3/E}(u_3)$.

3. $|u_1|_{Q_{11}} = 1$ and $|u_2|_{Q_{21}} = |u_3|_{Q_{31}} = 1/q$.

4. u_i is a square modulo Q_{ii} for $i = 1, 2$.

5. $u_1, h(u_2), h^2(u_3) > 0$.

The first and second conditions ensure that the map $\underline{f}_v$ is indeed a $\mathbb{Z}[G_v]$-module morphism. The next two conditions ensure that $\underline{f}_v$ represents the trivial element in $H^2(V_4, K_v^*)$ for $v = \mathfrak{q}_2, \mathfrak{q}_3$ and a generator in $H^2(V_4, K_{\mathfrak{q}}^*)$. This follows from Lemma 3. The last condition ensures that $\underline{f}_v$ represents the trivial element in $H^2(C_2, \mathbb{C}^*)$. Thus

Proposition 2. *If $\underline{f}_v : M \to U_S$ is defined by $\underline{f}_v(\alpha_i) = u_i$ for $i = 1, 2, 3$ where u_i's are S-units satisfying the above conditions, then the class $\beta \in \mathrm{Ext}^2_G(Y_v, U_S)$ corresponding to f_v satisfies*

$$\mathrm{inv}(G_v \cap G_w, w)(\beta) = \begin{cases} \frac{1}{\#(G_v \cap G_w)} & \text{if } w = v, \\ -\frac{1}{\#(G_v \cap G_w)} & \text{if } w = \mathfrak{m}, \\ 0 & \text{if } w \in S(v) \setminus \{v, \mathfrak{m}\}. \end{cases}$$

3.3.3 Case $v = \mathfrak{p}$

In this case $G_v = \{1, h, h^2\}$. Let L be the fixed field of G_v. Let $S(v) = \{\mathfrak{m}, \mathfrak{p}, \mathfrak{p}_2, \mathfrak{q}\} \cup \{$four primes above $l\} \cup \{$two infinite primes$\}$, a set of representatives for G_v-orbits. The group $G_v \cap G_w$ is trivial for $w \in S(v) \setminus \{\mathfrak{p}, \mathfrak{m}\}$, and equals G_v for $w = \mathfrak{p}, \mathfrak{m}$.

Consider the following G_v-resolution of $\mathbb{Z}$:

$$0 \to \mathbb{Z} \xrightarrow{\delta_3} \mathbb{Z}G_v \xrightarrow{\delta_2} \mathbb{Z}G_v \xrightarrow{\delta_1} \mathbb{Z} \to 0,$$

defined by $\delta_1(1) = 1, \delta_2(1) = 1 - h, \delta_3(1) = 1 + h + h^2$. Our aim therefore is to define $\underline{f}_v : \mathbb{Z} \to U_S$, and compute its image in the cohomology using the above resolution.

The construction of $\underline{f}_v$ requires an S-unit u_p satisfying

1. $u_p \in L$.

2. u_p is not congruent to ± 1 modulo the prime below $\mathfrak{p}$ in L.

The first condition is to make $\underline{f}_v$ an $\mathbb{Z}[G_v]$-module morphism, and second to ensure that u_p is not in $N_{K_{\mathfrak{p}}/L_P}(K_{\mathfrak{p}}^*)$ and hence $\underline{f}_v$ represents the generator in $H^2(G_v, K_v^*)$. This follows from the fact that an element $\alpha \in L_P^*$ is the norm of an element from K^* if and only if α is a cube module P, which is equivalent to α not being congruent to ± 1 modulo P. Thus,

Proposition 3. *If $\underline{f}_v : \mathbb{Z} \to U_S$ is defined by $\underline{f}_v(1) = u_p$, where u_p satisfies the conditions mentioned above, then the extension class $\beta \in \mathrm{Ext}^2_G(Y_v, U_S)$ corresponding to f_v satisfies*

$$\mathrm{inv}(G_v \cap G_w, w)(\beta) = \begin{cases} \frac{1}{\#(G_v \cap G_w)} & \text{if } w = v, \\ -\frac{1}{\#(G_v \cap G_w)} & \text{if } w = \mathfrak{m}, \\ 0 & \text{if } w \in S(v) \setminus \{v, \mathfrak{m}\}. \end{cases}$$

3.3.4 Case $v = \infty$

In this case $G_v = \{1, g_1\}$. Let $S(v) = \{\mathfrak{m}, \mathfrak{p}, \mathfrak{p}_2, \mathfrak{q}, \mathfrak{q}_2, \mathfrak{q}_3\} \cup \{\text{six primes over } l\} \cup \{\text{three infinite primes}\}$, a set of representatives for G_v-orbits. The group $G_v \cap G_w$ is nontrivial only for $w =$ primes over q and $w = \mathfrak{m}, v$.

Consider the following G_v-resolution of $\mathbb{Z}$:

$$0 \to \mathbb{Z} \xrightarrow{\delta_3} \mathbb{Z}G_v \xrightarrow{\delta_2} \mathbb{Z}G_v \xrightarrow{\delta_1} \mathbb{Z} \to 0,$$

defined by $\delta_1(1) = 1, \delta_2(1) = 1 - g_1, \delta_3(1) = 1 + g_1$. We need to construct $\underline{f}_v : \mathbb{Z} \to U_S$ such that its image in the cohomology groups are as desired.

So, the construction $\underline{f}_v$ requires a S-unit u_∞ satisfying

1. $u_\infty \in F$.
2. $|u_\infty|_{Q_i} = 1$ for $i = 1, 2, 3$.
3. u_∞ is a square modulo Q_1.
4. $u_\infty < 0$ and $\sigma(u_\infty) > 0$, where σ is the nontrivial element of $\mathrm{Gal}(F/E)$.

The first condition ensures that $\underline{f}_v$ is a $\mathbb{Z}[G_v]$-module morphism. The second and third conditions ensure that the element represented by $\underline{f}_v$ in $H^2(C_2, K_w^*)$ is trivial for $w = \mathfrak{q}, \mathfrak{q}_2, \mathfrak{q}_3$. The final condition ensures that $\underline{f}_v$ represents the nontrivial element in $H^2(C_2, K_v^*)$, and trivial element in $H^2(C_2, K_w^*)$ for infinite primes $w \neq v$. Thus,

Proposition 4. *If $\underline{f}_v : \mathbb{Z} \to U_S$ is defined by $\underline{f}_v(1) = u_\infty$, where u_∞ satisfies the conditions* *above, then the class $\beta \in \mathrm{Ext}^2_G(Y_v, U_S)$ corresponding to f_v satisfies*

$$\mathrm{inv}(G_v \cap G_w, w)(\beta) = \begin{cases} \frac{1}{\#(G_v \cap G_w)} & \textit{if } w = v, \\ -\frac{1}{\#(G_v \cap G_w)} & \textit{if } w = \mathfrak{m}, \\ 0 & \textit{if } w \in S(v) \setminus \{v, \mathfrak{m}\}. \end{cases}$$

3.3.5 Construction of the Tate sequence

PARI ([30]) was used to explicitly compute the S-units giving rise to the Tate sequence. Given that the S-units should satisfy certain conditions as in the previous subsection, the search for such units was done in the following fashion.

The units u_2 and u_3 have valuation $1/q$ with respect to the prime ideals Q_{21} and Q_{31} respectively. This essentially hints that we could choose them to be generators of these prime ideals (which are prinicipal). However, if u_2 and u_3 are chosen to be the generators as above, then the condition $N_{F_1/E}(u_1)N_{F_2/E}(u_2) = N_{F_3/E}(u_3)$ is not satisfied for any S-unit $u_1 \in F_1$. Therefore, we multiply u_3 by a generator of a prime ideal in F_3 lying above l. Then, one can find $u_1 \in F_1$ satisfying all the conditions required. Clearly, u_1 will also generate a prime ideal above l in F_1.

For u_p, one can choose it to be 2. However, to make the image of the map f bigger we choose a generator α_l of a prime ideal in L lying above l, and we set $u_p = 2\alpha_l$. This satisfies the required conditions.

Finally, none of the generators of the principal prime ideals in F_1 lying below the primes in S satisfy the conditions for u_∞. We therefore look at the products of prime ideals that are not principal. The class number of F_1 is 2 (by PARI) and therefore the product of any two ideals that are not principal is principal. One can find a prime ideal $\mathfrak{l}_F$ lying above l in F_1 such that a generator of the ideal $\mathfrak{l}_F\mathfrak{m}_F$ satisfies the required conditions, where $\mathfrak{m}_F$ is the prime ideal in F_1 lying above 2. We set u_∞ to be this generator.

With the help of PARI these S-units were found and in the following we shall explicitly write down one such possible set of S-units.

Let $\epsilon_7 = \zeta_p + \zeta_p^{-1}$. So, $E = \mathbb{Q}[\epsilon_7]$. Set

$$\theta_1 = 2\epsilon_7^2 - \epsilon_7 - 5,$$

$$\theta_2 = -3\epsilon_7^2 - 2\epsilon_7 + 3,$$

$$\theta_3 = \epsilon_7^2 + 3\epsilon_7 - 2.$$

Then θ_i generates a prime ideal above q for each $i = 1, 2, 3$. Let $v_1 = \sqrt{\theta_2/\theta_3}$. Since $\theta_2/\theta_3 > 0$, the extension $F = E[v_1]$ is real. Let $v_2 = \sqrt{\theta_3/\theta_1}$ and $v_3 = \sqrt{\theta_1/\theta_2}$. We shall express all the S-units in terms of ϵ_7 and v_i's. The following is a set of S-units satisfying the required conditions:

- $u_1 = (3\epsilon_7^2 + 7\epsilon_7 + 2) + (2\epsilon_7^2 + 7\epsilon_7 + 5)v_1,$
- $u_2 = (1472\epsilon_7^2 - 1182\epsilon_7 - 816) + (-1697\epsilon_7^2 + 1359\epsilon_7 + 943)v_2,$
- $u_3 = (782\epsilon_7^2 + 435\epsilon_7 - 1756) + (901\epsilon_7^2 + 503\epsilon_7 - 2021)v_3,$
- $u_p = 8 + (6\epsilon_7^2 + 4\epsilon_7 - 6)v_1 + (-2\epsilon_7^2 - 6\epsilon_7 + 4)v_2 + (4\epsilon_7^2 - 2\epsilon_7 - 10)v_3,$
- $u_\infty = (5\epsilon_7^2 + 7\epsilon_7 - 2) + (-2\epsilon_7^2 - 7\epsilon_7 - 5)v_1.$

This gives a map $f : X_S(-2) \to U_S$ which represents the canonical class in $\mathrm{Ext}^2_G(X_S, U_S)$. Now, with the above setting, we can find a S-unit u_0, again using PARI, such that the map $\tilde{f} : X_S(-2) \oplus N \to U_S$, where $N \simeq \mathbb{Z}[G]$, extending f by $\tilde{f}((0,1)) = u_0$ is surjective. The image of f has $\mathbb{Z}$-rank 21 while the $\mathbb{Z}$-rank of U_S is 25. Further, the valuation with respect to $\mathfrak{p}$ of all the S-units that are chosen so far is 1. Thus, we choose u_0 such that its valuation with respect to $\mathfrak{p}$ is $1/p$. This will make the rank equal 25. Further the following choice of such a unit also makes the map $\tilde{f}$ surjective:

- $u_0 = (4983/2\epsilon_7^2 + 5597\epsilon_7 + 3991/2) + (-5735/2\epsilon_7^2 - 6444\epsilon_7 - 2300)v_1 + (2322\epsilon_7^2 + 10435/2\epsilon_7 + 1862)v_2 + (-5087/2\epsilon_7^2 - 11435/2\epsilon_7 - 2043)v_3.$

Note that by the construction of $\tilde{f}$, it induces an isomorphism on the cohomology groups corresponding to $X_S(-2)$ and U_S. Thus, if P denotes the kernel of $\tilde{f}$, then P is cohomologically trivial and torsion free. This shows that P is projective. But, by the Proposition 1 we know that every projective $\mathbb{Z}[A_4]$ module is free. Hence P is free. Thus, we have our Tate sequence

$$0 \to U_S \to (A \oplus N)/P \to B \to X_S \to 0.$$

3.4 The main result

Our aim of this section is to prove the Conjecture 2 in the particular case of $K/\mathbb{Q}$ that we are considering. This we shall achieve by computing the leading coefficients of the L-functions and equating that with an element coming from the Tate sequence.

3.4.1 Leading coefficients of L-functions

Let $k = \mathbb{Q}$ and $K/\mathbb{Q}$ be the Galois extension defined in the earlier section, with Galois group $G = \mathrm{Gal}(K/\mathbb{Q}) \simeq A_4$. There are four irreducible complex characters of A_4. The character table is below.

	1	g_1	h	h^2
χ_0	1	1	1	1
χ_1	1	1	ζ_3	ζ_3^2
χ_1'	1	1	ζ_3^2	ζ_3
χ_2	3	-1	0	0

The character χ_0 is the trivial character. There are two other abelian characters χ_1 and χ_1'. The last character χ_2 corresponds to the 3-dimensional irreducible representation. For χ_0, we have $L(\chi_0, 0) = \zeta_{\mathbb{Q}}(s)$ and therefore $L^*(\chi_0, 0) = -1/2$ and $L_S^*(\chi_0, 0) = -\frac{1}{2}(\log m)(\log p)(\log q)(\log l)$.

The character χ_1 is abelian and hence factors through $G^{\mathrm{ab}} \simeq C_3$. We have (cf. [37])

$$L^*(\chi_1, 0) = -\frac{1}{2}\sum_{a=1}^{6} \log|1 - \zeta_7^a|\underline{\chi}_1(a),$$

where $\zeta_7 = e^{2\pi i/7}$ is a primitive seventh root of unity and $\underline{\chi}_1$ is defined on $(\mathbb{Z}/7\mathbb{Z})^\times$ by inflating via $(\mathbb{Z}/7\mathbb{Z})^\times \to C_3 \xrightarrow{\chi_1} \mathbb{C}^\times$. Let $\alpha_7 = (1-\zeta_7)(1-\zeta_7^{-1}) = 2 - \epsilon_7 \in E$. Then α_7 generates the unique prime ideal in E above p. In terms of α_7 we have

$$\begin{aligned} L^*(\chi_1, 0) &= -\frac{1}{2}(\log|\alpha_7| + \zeta_3 \log|h(\alpha_7)| + \zeta_3^2 \log|h^2(\alpha_7)|) \\ &= -\frac{1}{2}\left(\log|\frac{\alpha_7}{h(\alpha_7)}| + \zeta_3^2 \log|\frac{h^2(\alpha_7)}{h(\alpha_7)}|\right). \end{aligned}$$

Let $w_1 = (\epsilon_7 + 1)^{-1}$ and $w_2 = \epsilon_7$. Then, $\{w_1, w_2\}$ generate the unit group in E. Further, $\frac{\alpha_7}{h(\alpha_7)} = w_1^2$ and $\frac{h(\alpha_7)}{h^2(\alpha_7)} = w_2^2$. The above equation reduces to

$$L^*(\chi_1, 0) = -(\log|w_1| - \zeta_3^2 \log|w_2|).$$

Further, we have $L_S^*(\chi_1, 0) = (1 - \zeta_3^2)(\log l)(\log q)L^*(\chi_1, 0)$.

The character χ_2 decomposes as $\chi_2 = \mathrm{Ind}_{C_2}^G 1_{C_2} - \mathrm{Ind}_{V_4}^G 1_{V_4}$, where $C_2 = \mathrm{Gal}(K/F)$ and $V_4 = \mathrm{Gal}(K/E)$. Thus, $L(\chi_2, s) = \frac{L(1_{C_2}, s)}{L(1_{V_4}, s)} = \frac{\zeta_F(s)}{\zeta_E(s)}$. Thus, again by class number formula, we get $L^*(\chi_2, 0) = \frac{h_F R_F}{h_E R_E}$, where h_L and R_L denote the class number and the regulator of L respectively. We know that $h_E = 1$. Further, we have $h_F = 2$, by PARI.

Note that E is a totally real Galois extension of $\mathbb{Q}$ and thus, there are three real embeddings in E say $\sigma_1 = \mathrm{id}, \sigma_2$ and σ_3. Hence the unit group is of $\mathbb{Z}$-rank 2. Let w_1 and w_2 be the $\mathbb{Z}$-generators of this group as defined above. It is easy to verify that F has two real embeddings and four complex embeddings. Thus, the unit group of F will be of $\mathbb{Z}$-rank 3. One can choose a unit w_3 in F such that $\{w_1, w_2, w_3\}$ is a basis of the unit group in F. This follows from the fact that $F \neq E[\sqrt(u)]$ for any unit $u \in E$. Let id, σ_r be the two real embeddings of F and let $\sigma_c, \sigma_{c'}$ be two complex embeddings. Then,

$$R_E = \left| \det \begin{pmatrix} \log|w_1| & \log|\sigma_2(w_1)| \\ \log|w_2| & \log|\sigma_2(w_2)| \end{pmatrix} \right|,$$

$$R_F = \left| \det \begin{pmatrix} \log|w_1| & \log|\sigma_r(w_1)| & 2\log|\sigma_c(w_1)| \\ \log|w_2| & \log|\sigma_r(w_2)| & 2\log|\sigma_c(w_2)| \\ \log|w_3| & \log|\sigma_r(w_3)| & 2\log|\sigma_c(w_3)| \end{pmatrix} \right|.$$

By assumption, F is a real field, and therefore σ_r is an element of $\mathrm{Gal}(F/E)$. Thus, $\sigma_r(w_i) = w_i$ for $i = 1, 2$. Also, we can choose σ_c such that $\sigma_c(w_i) = \sigma_2(w_i)$ for $i = 1, 2$. Thus, $R_F = 2|\log|\frac{w_3}{\sigma_r(w_3)}||R_E$, and therefore we have $L^*(\chi_2, 0) = 4|\log|\frac{w_3}{\sigma_r(w_3)}||$. Further, $L_S^*(\chi_2, 0) = 8(\log p)(\log l)^3|\log|\frac{w_3}{\sigma_r(w_3)}||$.

3.4.2 Determinants

From the construction in the previous section we have a commutative diagram of exact sequences:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & & \\
 & P \arrow[r] \arrow[d] & \lambda(P) \arrow[d] & & & \\
0 \arrow[r] & X_S(-2) \oplus N \arrow[r, "\lambda"] \arrow[d, "h"] & A \oplus N \arrow[r, "\theta"] \arrow[d] & B \arrow[r, "\mu"] \arrow[d, equal] & X_S \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & U_S \arrow[r] \arrow[d] & A_S \arrow[r] \arrow[d] & B_S \arrow[r] & X_S \arrow[r] & 0 \\
 & 0 & 0 & & &
\end{tikzcd}$$

where $A_S := (\frac{A \oplus N}{\lambda(P)})$ and $B_S := B$. Then we have

$$\begin{aligned}
\mathrm{Det}(A_{S,\mathbb{R}}) \otimes \mathrm{Det}^{-1}(B_{S,\mathbb{R}}) &\simeq \mathrm{Det}^{-1}(\lambda(P)_{\mathbb{R}}) \otimes \mathrm{Det}(A_{\mathbb{R}} \oplus N_{\mathbb{R}}) \otimes \mathrm{Det}^{-1}(B_{\mathbb{R}}) \\
&\simeq \mathrm{Det}^{-1}(\lambda(P)_{\mathbb{R}}) \otimes \mathrm{Det}(X_S(-2)_{\mathbb{R}} \oplus N_{\mathbb{R}}) \otimes \\
&\quad \mathrm{Det}^{-1}(X_{S,\mathbb{R}}) \\
&\simeq \mathrm{Det}^{-1}(\lambda(P)_{\mathbb{R}}) \otimes \mathrm{Det}(P_{\mathbb{R}}) \otimes \mathrm{Det}(U_{S,\mathbb{R}}) \otimes \\
&\quad \mathrm{Det}^{-1}(X_{S,\mathbb{R}}) \\
&\simeq \mathrm{Det}(0) \simeq \zeta(\mathbb{R}[G]).
\end{aligned}$$

The first isomorphism follows from the construction of A_S, the second from the exactness of the middle row, the third via the map h and the last isomorphism via the maps λ and R_S. The composition of all these isomorphisms is the map induced by $\psi_S : A_{S,\mathbb{R}} \simeq B_{S,\mathbb{R}}$.

Note that the image of $(A_S, B_S; \psi_S)$ in $K_0(\mathbb{Z}[G])$ is zero. Thus, one can look for the inverse image of this element in $K_1(\mathbb{R}[G])$ via δ. After choosing $\mathbb{Z}[G]$ bases for A, N, B and P, we can find a generator for

$$\mathrm{Det}^{-1}(\lambda(P)_{\mathbb{R}}) \otimes \mathrm{Det}(A_{\mathbb{R}} \oplus N_{\mathbb{R}}) \otimes \mathrm{Det}^{-1}(B_{\mathbb{R}})$$

arising from these bases. The image of this in $\zeta(\mathbb{R}[G])$ under the above isomorphism is in the inverse image $\delta^{-1}((A_S, B_S; \psi_S))$. Our aim now is to compute this element in the inverse image.

First we shall choose bases for A, B and N. The exact sequence

$$0 \to X_S(-2) \oplus N \xrightarrow{\lambda} A \oplus N \xrightarrow{\theta} B \xrightarrow{\mu} X_S \to 0$$

splits into different prime components as

$$0 \to Y_v(-2) \xrightarrow{\lambda_v} A_v \xrightarrow{\theta_v} B_v \xrightarrow{\mu_v} Y_v \to 0$$

and of course $\lambda|_N : N \to N$ is the identity map. We shall fix the generators for A, B and N as follows:

- For $v = \infty$, let $A_v = \mathbb{Z}[G]a_1, B_v = \mathbb{Z}[G]b_1$ with $\lambda_v(1) = (1 + g_1)a_1, \theta_v(a_1)$ $= (1 - g_1)b_1, \mu_v(b_1) = 1$.
- For $v = p$, let $A_v = \mathbb{Z}[G]a_2, B_v = \mathbb{Z}[G]b_2$ with $\lambda_v(1) = (1 + h + h^2)a_2, \theta_v(a_2)$ $= (1 - h)b_2, \mu_v(b_2) = 1$.
- For $v = q$, let $A_v = \mathbb{Z}[G]a_3 \oplus \mathbb{Z}[G]a_4, B_v = \mathbb{Z}[G]b_3$ with $\lambda_v(\alpha_1) = (-1 - g_1)a_3$ $+ (1 + g_1)a_4, \lambda_v(\alpha_2) = (1 + g_2)a_3, \lambda_v(\alpha_3) = (1 + g_3)a_4, \theta_v(a_3) = (1 - g_2)b_3, \theta_v(a_4)$ $= (1 - g_3)b_3, \mu_v(b_3) = 1$.
- For $v = l$, $A_v = 0$ and $B_v = \mathbb{Z}[G]b_4$ with $\mu_v(b_4) = 1$.
- Finally, let $N = \mathbb{Z}[G]a_5$ with $\tilde{f}(a_5) = u_0$.

We then choose a generator c for $P \simeq \mathbb{Z}[G]$, and the image of this generator in $X_v(-2)$ and N is given by:

$$c = (c_\infty, c_p, c_q, c_N) \in \mathrm{Ind}_{C_2}^G \mathbb{Z} \oplus \mathrm{Ind}_{C_3}^G \mathbb{Z} \oplus \mathrm{Ind}_{V_4}^G M \oplus N,$$

where

$$\begin{aligned} c_\infty &= (-3 - g_2 + 5h + 5hg_2 - 3h^2 - 4h^2 g_2), \\ c_p &= 0, \\ c_q &= (-5 - 5g_1 + 3h + 4hg_1 + 3h^2 + h^2 g_1)(\alpha_3 - \alpha_2) + (h^2 - h^2 g_1)\alpha_2, \\ c_N &= (g_1 + g_2 + g_3)(1 - h^2). \end{aligned}$$

Finding such a c is a problem in linear algebra, and PARI was used to carry out the computations.

We set $z = (\wedge_{i=1}^5 a_i) \in \mathrm{Det}(A \oplus N)$, $z' = [\wedge_{i=1}^4 b_i]^{-1} \in \mathrm{Det}^{-1}(B)$ and $z'' = [\lambda(c)]^{-1} \in \mathrm{Det}^{-1}(\lambda(P))$. We shall compute the image of $z'' \otimes z \otimes z'$ in $\zeta(\mathbb{R}[G])$.

We now look at the idempotents of $\mathbb{R}[G]$. Let $e_0 = \frac{1}{12}\sum_{g\in G} g, e_1 = \frac{1}{12}(2e - h - h^2)(e + g_1 + g_2 + g_3)$, and $e_2 = \frac{1}{4}(3e - g_1 - g_2 - g_3)$. Under the natural isomorphism $\rho : \mathbb{R}[G] \to \mathbb{R} \times \mathbb{C} \times M_3(\mathbb{R}) =: R_0 \times R_1 \times R_2$, these idempotents map to $(1,0,0), (0,1,0)$ and $(0,0,\mathrm{id})$ respectively. Thus, we have $e_0 + e_1 + e_2 = 1$. Further, for a $\mathbb{Z}[G]$-module M, we set $M_i := e_i M_{\mathbb{R}}$, which we shall regard as a module over R_i. For a morphism $f \in \mathrm{Hom}_{\mathbb{R}[G]}(M, N)$, let f_i denote the corresponding element in $\mathrm{Hom}_{R_i}(M_i, N_i)$.

3.4.3 The idempotent e_0

First note that the map $\theta_0 : A_0 \oplus N_0 \to B_0$ is the zero map. Therefore, λ_0 and μ_0 are isomorphisms. The module $A_0 \oplus N_0$ is generated by $\{e_0 a_i\}_{i=1}^5$, B_0 is generated by $\{e_0 b_j\}_{j=1}^4$. The image of $e_0(z \otimes z') = (\wedge_{i=1}^5 e_0 a_i) \otimes [\wedge_{j=1}^4 e_0 b_j]^{-1}$ in $\mathrm{Det}(X_S(-2)_0 \oplus N_0) \otimes \mathrm{Det}^{-1}(X_{S,0})$ is

$$\overline{z}_0 \otimes \overline{z'}_0 := \frac{1}{24}(\wedge_{i=1}^5 e_0 d_i) \otimes [e_0(\infty - \mathfrak{m}) \wedge e_0(\mathfrak{p} - \mathfrak{m}) \wedge e_0(\mathfrak{q} - \mathfrak{m}) \wedge e_0(\mathfrak{l} - \mathfrak{m})]^{-1},$$

where in $X_S(-2)_{\mathbb{R}} \oplus N_{\mathbb{R}} \simeq (\mathrm{Ind}_{C_2}^G \mathbb{R}) \oplus (\mathrm{Ind}_{C_3}^G \mathbb{R}) \oplus (\mathrm{Ind}_{V_4}^G M_{\mathbb{R}}) \oplus N_{\mathbb{R}}$, the elements d_i's are defined by

- $d_1 = (1, 0, 0, 0)$
- $d_2 = (0, 1, 0, 0)$,
- $d_3 = (0, 0, \alpha_2, 0)$,
- $d_4 = (0, 0, \alpha_3, 0)$,

- $d_5 = (0,0,0,a_5)$.

Now, the map $(R_{S,0} \circ \tilde{f}_0) : X_S(-2)_0 \oplus N_0 \to X_{S,0}$ is given by

$$e_0 d_i \mapsto -e_0 \sum_{v \in S} \log |\tilde{f}_0(d_i)|_v (v - \mathfrak{m}).$$

So, the images of $e_0 d_i$ are:

- $e_0 d_1 \mapsto -\log |N(u_\infty)| e_0(\infty - \mathfrak{m}) + 2(\log l) e_0(\mathfrak{l} - \mathfrak{m})$,
- $e_0 d_2 \mapsto -\log |N(u_p)| e_0(\infty - \mathfrak{m}) + 3(\log l) e_0(\mathfrak{l} - \mathfrak{m})$,
- $e_0 d_3 \mapsto -\log |N(u_2)| e_0(\infty - \mathfrak{m}) + 2(\log q) e_0(\mathfrak{q} - \mathfrak{m})$,
- $e_0 d_4 \mapsto -\log |N(u_3)| e_0(\infty - \mathfrak{m}) + 2(\log l) e_0(\mathfrak{l} - \mathfrak{m}) + 2(\log q) e_0(\mathfrak{q} - \mathfrak{m})$,
- $e_0 d_5 \mapsto -\log |N(u_0)| e_0(\infty - \mathfrak{m}) + (\log p) e_0(\mathfrak{p} - \mathfrak{m})$,

where $N = N_{K/\mathbb{Q}}$ is the norm map from K to $\mathbb{Q}$.

The module P_0 is generated by c_0 whose image in $X_S(-2)_0 \oplus N_0$ is $e_0(-d_1 - d_3 + d_4)$, respectively. Thus,

$$\begin{aligned}
\overline{z}_0 &= \frac{1}{24}(\wedge_{i=1}^5 e_0 d_i) \\
&= -\frac{1}{24}(\mathrm{im}(c_0) \wedge e_0 d_1 \wedge e_0 d_2 \wedge e_0 d_3 \wedge e_0 d_5) \\
&\mapsto -\frac{1}{24}(e_0 c) \otimes (\wedge_{i=1,2,3,5}(R_{S,0} \circ \tilde{f}_0)(e_0 d_i)) \\
&= \frac{1}{24}(e_0 c) \otimes 2(\log \alpha)(\log l)(\log q)(\log p)(\overline{z_0'})^{-1},
\end{aligned}$$

where $\alpha = |N(\frac{u_p^2}{u_\infty^3})|$. But this equals m^6 and thus, the image of $e_0(z'' \otimes z \otimes z')$ in $\zeta(\mathbb{R}[G])$ is

$$\frac{1}{2}(\log m)(\log l)(\log q)(\log p) e_0 = -e_0 L_S^*(\chi_0, 0) = -e_0 L_S^*(\chi_0, 0)^{\#}.$$

3.4.4 The idempotent e_1

For each $v \in S$ consider the exact sequence

$$0 \to Y_v(-2)_1 \overset{\lambda_{v,1}}{\to} A_{v,1} \overset{\theta_{v,1}}{\to} B_{v,1} \overset{\mu_{v,1}}{\to} Y_{v,1} \to 0.$$

The maps $\theta_{\infty,1}, \theta_{\mathfrak{q},1}$ and $\theta_{\mathfrak{l},1}$ are zero maps while $\theta_{\mathfrak{p},1}$ is an isomorphism given by $\theta_{\mathfrak{p},1}(e_1a_2) = (1-\zeta_3)e_1b_2$. The module $A_1 \oplus N_1$ is generated by $\{e_1a_i\}_{i=1}^5$ and B_1 by $\{e_1b_j\}_{j=1}^4$. The image of $e_1(z\otimes z') = (\wedge_{i=1}^5 e_1a_i)\otimes[\wedge_{i=1}^4 e_1b_j]^{-1}$ in $\mathrm{Det}(X_S(-2)_1\oplus N_1)\otimes\mathrm{Det}^{-1}(X_{S,1})$ is

$$\overline{z}_1\otimes\overline{z'}_1 := \frac{1-\zeta_3}{8}(\wedge_{i=1}^4 e_1d_i)\otimes[e_1(\infty-\mathfrak{m})\wedge e_1(\mathfrak{q}-\mathfrak{m})\wedge e_1(\mathfrak{l}-\mathfrak{m})]^{-1},$$

where in $X_S(-2)_{\mathbb{R}}\oplus N_{\mathbb{R}}\simeq \mathrm{Ind}_{C_2}^G\mathbb{R}\oplus\mathrm{Ind}_{C_3}^G\mathbb{R}\oplus\mathrm{Ind}_{V_4}^G M_{\mathbb{R}}\oplus N_{\mathbb{R}}$, the elements d_i's are defined by

- $d_1 = (1,0,0,0)$,
- $d_2 = (0,0,\alpha_2,0)$,
- $d_3 = (0,0,\alpha_3,0)$,
- $d_4 = (0,0,0,a_5)$.

Now, the map $(R_{S,1}\circ\tilde{f}_1): X_S(-2)_1\oplus N_1\to X_{S,1}$ is given by

$$e_1d_i\mapsto -e_1\sum_{v\in S}\log|\tilde{f}_1(d_i)|_v(v-\mathfrak{m}).$$

So, the images of e_1d_i are:

- $e_1d_1\mapsto -e_1(\sum\log|u_\infty^{g^{-1}}|g)(\infty-\mathfrak{m}) + 2(\log l)e_1(\mathfrak{l}-\mathfrak{m})$,
- $e_1d_2\mapsto -e_1(\sum\log|u_2^{g^{-1}}|g)(\infty-\mathfrak{m}) + 2(\log q)e_1(\mathfrak{q}-\mathfrak{m})$,
- $e_1d_3\mapsto -e_1(\sum\log|u_3^{g^{-1}}|g)(\infty-\mathfrak{m}) + 2(\log l)e_1(\mathfrak{l}-\mathfrak{m}) + 2(\log q)e_1(\mathfrak{q}-\mathfrak{m})$,
- $e_1d_4\mapsto -e_1(\sum\log|u_0^{g^{-1}}|g)(\infty-\mathfrak{m})$.

For $u\in K$, we have $e_1(\sum\log|u^{g^{-1}}|g) = (\log|N_{K/E}(u)|+\zeta_3\log|h^2(N_{K/E}(u))|+\zeta_3^2\log|h(N_{K/E}(u))|)e_1$, when considered as an element of R_1, under the projection $\mathbb{R}[G]\to R_1$. Let

$$\alpha(u) := (\log|N_{K/E}(u)| + \zeta_3\log|h^2(N_{K/E}(u))| + \zeta_3^2\log|h(N_{K/E}(u))|).$$

The module P_1 is generated by e_1c whose image in $X_S(-2)_1 \oplus N_1$ is $e_1((3+17\zeta_3)d_1 + (17+3\zeta_3)d_2 + (-17-3\zeta_3^2)d_3 + (6+3\zeta_3)d_4)$. Therefore,

$$\begin{aligned}
\overline{z}_1 &= \frac{(1-\zeta_3)}{8}(\wedge_{i=1}^4 e_1 d_i) \\
&= \frac{(1-\zeta_3)}{8(3+17\zeta_3)}(\mathrm{im}(e_1c) \wedge e_1d_2 \wedge e_1d_3 \wedge e_1d_4) \\
&\mapsto \frac{(1-\zeta_3)}{8(3+17\zeta_3)}(e_1c) \otimes (\wedge_{i=2,3,4}(R_{S,1} \circ \tilde{f}_1)(e_1d_i)) \\
&= \frac{(1-\zeta_3)}{8(3+17\zeta_3)}(e_1c) \otimes -4\alpha(u_0)(\log l)(\log q)(\overline{z_1'})^{-1}.
\end{aligned}$$

To calculate $\alpha(u_0)$ we consider the element $\beta_p = N_{K/E}(u_0) \in E$. We have that $\frac{\beta_p}{h(\beta_p)}$ and $\frac{h(\beta_p)}{h^2(\beta_p)}$ are units in E. Thus we can write these in terms of the basis $\{w_1, w_2\}$ as follows:

$$\frac{\beta_p}{h(\beta_p)} = w_1^{-28}w_2^6, \frac{h(\beta_p)}{h^2(\beta_p)} = w_1^{-6}w_2^{-34}.$$

Hence,

$$\begin{aligned}
\alpha(u_0) &= (\log|\beta_p| + \zeta_3 \log|h^2(\beta_p)| + \zeta_3^2 \log|h(\beta_p)|) \\
&= \left(\log|\frac{\beta_p}{h(\beta_p)}| - \zeta_3 \log|\frac{h(\beta_p)}{h^2(\beta_p)}|\right) \\
&= -28\log|w_1| + 6\log|w_2| + 6\zeta_3\log|w_1| + 34\zeta_3\log|w_2| \\
&= -\zeta_3^2(6+34\zeta_3)(\log|w_1| - \zeta_3\log|w_2|) \\
&= \zeta_3^2(6+34\zeta_3)L^*(\chi_1,0)^{\#}.
\end{aligned}$$

Since $L_S^*(\chi_1,0)^{\#} = (1-\zeta_3)(\log l)(\log q)L^*(\chi_1,0)^{\#}$ it follows that the image of $e_1(z'' \otimes z \otimes z')$ in $R_2^{\times}$ is $-\zeta_3^2 e_1 L_S^*(\chi_1,0)^{\#}$.

3.4.5 The idempotent e_2

Let $e_3 = \frac{1}{4}(e + g_1 - g_2 - g_2)$. Then e_3 is an indecomposable idempotent of $\mathbb{R}[G]$ which lies in $R_2 = M_3(\mathbb{R})$. If V is a finitely generated projective module over R_2, then we can consider the e_3V as a module over $\zeta(R_2) \simeq \mathbb{R}$. Then, we have that the determinant of V over R_2 is the same as the determinant of e_3V over $\zeta(R_2)$.

Now, for each $v \in S_0$, consider

$$0 \to Y_v(-2)_3 \stackrel{\lambda_{v,3}}{\to} A_{v,3} \stackrel{\theta_{v,3}}{\to} B_{v,3} \stackrel{\mu_{v,3}}{\to} Y_{v,3} \to 0,$$

which is an exact sequence of $\mathbb{R}$-modules. These sequences are defined in terms of the bases of A_3, B_3 as follows:

$v = \infty$

$$\begin{aligned}
A_{v,3} &\simeq \mathbb{R}^3 \text{ with basis } \{e_3 h^i a_1\}_{i=0,1,2} \\
B_{v,3} &\simeq \mathbb{R}^3 \text{ with basis } \{e_3 h^i b_1\}_{i=0,1,2} \\
Y_{v,3} &\simeq Y_v(-2)_3 \simeq \mathbb{R} \text{ generated by } e_3 \\
\lambda_{v,3}(e_3) &= 2e_3 a_1 \\
\theta_{v,3}(\alpha e_3 a_1 + \beta e_3 h a_1 + \gamma e_3 h^2 a_1) &= 2(\beta e_3 h b_1 + \gamma e_3 h^2 b_1) \\
\mu_{v,3}(\alpha e_3 b_1 + \beta e_3 h b_1 + \gamma e_3 h^2 b_1) &= \alpha e_3
\end{aligned}$$

$v = \mathfrak{p}$

$$\begin{aligned}
A_{v,3} &\simeq \mathbb{R}^3 \text{ with basis } \{e_3 h^i a_2\}_{i=0,1,2} \\
B_{v,3} &\simeq \mathbb{R}^3 \text{ with basis } \{e_3 h^i b_2\}_{i=0,1,2} \\
Y_{v,3} &\simeq Y_v(-2)_3 \simeq \mathbb{R} \text{ generated by } e_3 \\
\lambda_{v,3}(e_3) &= (e_3 a_2 + e_3 h a_2 + e_3 h^2 a_2) \\
\theta_{v,3}(\alpha e_3 a_1 + \beta e_3 h a_1 + \gamma e_3 h^2 a_1) &= (\alpha - \gamma) e_3 b_2 + (\beta - \alpha) e_3 h b_2 \\
&\quad + (\gamma - \beta) e_3 h^2 b_2 \\
\mu_{v,3}(\alpha e_3 b_2 + \beta e_3 h b_2 + \gamma e_3 h^2 b_2) &= (\alpha + \beta + \gamma) e_3
\end{aligned}$$

$v = \mathfrak{q}$

$$\begin{aligned}
A_{v,3} &\simeq \mathbb{R}^6 \text{ with basis } \{e_3 h^j a_i\}_{i=3,4,j=0,1,2} \\
B_{v,3} &\simeq \mathbb{R}^3 \text{ with basis } \{e_3 h^i b_3\}_{i=0,1,2} \\
Y_v(-2)_3 &\simeq \mathbb{R}^3 \text{ with basis } \{e_3 h^i \alpha_{i+1}\}_{i=0,1,2} \\
Y_{v,3} &= 0 \\
\lambda_{v,3}(\alpha e_3 \alpha_1 + \beta e_3 h \alpha_2 + \gamma e_3 h^2 \alpha_3) &= 2\alpha e_3(-a_3 + a_4) + 2\beta e_3 h a_3 \\
&\quad +2\gamma e_3 h^2 a_4 \\
\theta_{v,3}(\alpha e_3 a_3 + \beta e_3 h a_3 + \gamma e_3 h^2 a_3) &= 2(\alpha e_3 b_3 + \gamma e_3 h^2 b_3) \\
\theta_{v,3}(\alpha' e_3 a_4 + \beta' e_3 h a_4 + \gamma' e_3 h^2 a_4) &= 2(\alpha' e_3 b_3 + \beta' e_3 h b_3) \\
\mu_{v,3} &= 0
\end{aligned}$$

$v = \mathfrak{l}$

$$\begin{aligned}
A_v &= 0 \\
B_{v,3} &\simeq \mathbb{R}^3 \text{ with basis } \{e_3 h^i b_4\}_{i=0,1,2} \\
Y_{v,3} &\simeq \mathbb{R}^3 \text{ with basis } \{e_3 h^i\}_{i=0,1,2} \\
\mu_{v,3}(\alpha e_3 b_4 + \beta e_3 h b_4 + \gamma e_3 h^2 b_4) &= \alpha e_3 + \beta e_3 h + \gamma e_3 h^2
\end{aligned}$$

Now, let $x_{v,3}$ and $y_{v,3}$ be generators of $\mathrm{Det}(A_{v,3})$ and $\mathrm{Det}(B_{v,3})$ respectively, arising from the bases mentioned above. Also, let $x_{N,3}$ be the generator of $\mathrm{Det}(N_3)$ with respect to the basis mentioned earlier. Let

$$(z_3 \otimes z_3') \quad := \quad (x_{\infty,3} \wedge x_{p,3} \wedge x_{q,3} \wedge x_{1,3} \wedge x_{N,3}) \otimes [y_{\infty,3} \wedge y_{p,3} \wedge y_{q,3} \wedge y_{l,3}]^{-1}.$$

This is a generator of $\mathrm{Det}(A_3 \oplus N_3) \otimes \mathrm{Det}^{-1}(B_3)$.

Let $\overline{z}_3 \otimes \overline{z'}_3$ be the image of $z_3 \otimes z_3'$ in $\mathrm{Det}(X_S(-2)_3 \oplus N_3) \otimes \mathrm{Det}^{-1}(X_{S,3})$. Then, $\overline{z}_3 = 2(\wedge_{i=1}^8 e_3 d_i)$ where d_i's are elements of $X_S(-2)_{\mathbb{R}} \oplus N_{\mathbb{R}} \simeq \mathrm{Ind}_{C_2}^G \mathbb{R} \oplus \mathrm{Ind}_{C_3}^G \mathbb{R} \oplus \mathrm{Ind}_{V_4}^G M_{\mathbb{R}} \oplus N_{\mathbb{R}}$ defined by:

- $d_1 = (1, 0, 0, 0)$,

- $d_2 = (0, 1, 0, 0)$,
- $d_3 = (0, 0, \alpha_1, 0)$,
- $d_4 = (0, 0, h^2\alpha_2, 0)$,
- $d_5 = (0, 0, h\alpha_3, 0)$,
- $d_6 = (0, 0, 0, a_5)$,
- $d_7 = (0, 0, 0, ha_5)$,
- $d_8 = (0, 0, 0, h^2a_5)$.

Further, $\overline{z'}_3 = [e_3(\infty - \mathfrak{m}) \wedge e_3(\mathfrak{p} - \mathfrak{m}) \wedge \wedge e_3(\mathfrak{l} - \mathfrak{m}) \wedge e_3h(\mathfrak{l} - \mathfrak{m}) \wedge e_3h^2(\mathfrak{l} - \mathfrak{m})]^{-1}$.

The map $(R_{S,3} \circ \tilde{f}_3) : X_S(-2)_3 \oplus N_3 \to X_{S,3}$ is given by

$$e_3d_i \mapsto -e_3 \sum_{v \in S} \log |\tilde{f}_3(d_i)|_v(v - \mathfrak{m}).$$

So, the images of e_3d_i are:

- $e_3d_1 \mapsto \beta(u_\infty)e_3(\infty - \mathfrak{m}) + 2(\log l)e_3(\mathfrak{l} - \mathfrak{m})$,
- $e_3d_2 \mapsto \beta(u_p)e_3(\infty - \mathfrak{m}) + (\log l)e_3(1 + h + h^2)(\mathfrak{l} - \mathfrak{m})$,
- $e_3d_3 \mapsto \beta(u_1)e_3(\infty - \mathfrak{m}) + 2(\log l)e_3(\mathfrak{l} - \mathfrak{m})$,
- $e_3d_4 \mapsto \beta(h^2(u_2))e_3(\infty - \mathfrak{m})$,
- $e_3d_5 \mapsto \beta(h(u_3))e_3(\infty - \mathfrak{m}) + 2(\log l)e_3h(\mathfrak{l} - \mathfrak{m})$,
- $e_3d_6 \mapsto \beta(u_0)e_3(\infty - \mathfrak{m}) + (\log p)e_3(\mathfrak{p} - \mathfrak{m})$,
- $e_3d_7 \mapsto \beta(h(u_0))e_3(\infty - \mathfrak{m}) + (\log p)e_3h(\mathfrak{p} - \mathfrak{m})$,
- $e_3d_8 \mapsto \beta(h^2(u_0))e_3(\infty - \mathfrak{m}) + (\log p)e_3h^2(\mathfrak{p} - \mathfrak{m})$,

where $\beta(u) = -\log |\frac{u^{g_2} \cdot u^{g_3}}{u \cdot u^{g_1}}|$.

The module P_3 is generated by $e_3h^jc, j = 0, 1, 2$. The images of these in $X_S(-2)_3 \oplus N_3$ is given by

- $e_3c \mapsto e_3(-2d_1 + d_4 - 2d_5 - d_6 + d_8)$,

- $e_3hc \mapsto e_3(d_1 - d_5 + d_6 - d_7)$,
- $e_3h^2c \mapsto e_3(d_7 - d_8)$.

Let z_3'' denote the generator of $\mathrm{Det}(P_3)$ with respect to the above basis. Then, with these values one has

$$\begin{aligned}\overline{z}_3 &= -2(\wedge_j \mathrm{im}(e_3h^jc) \wedge e_3d_2 \wedge e_3d_3 \wedge e_3d_4 \wedge e_3d_5 \wedge e_3d_7) \\ &\mapsto 8([z_3'']^{-1}) \otimes \beta(h^2(u_2))(\log l)^3(\log p)[\overline{z'}_3]^{-1}.\end{aligned}$$

The unit u_2 is the generator of an ideal in F_2 lying above q. The choice of this gives $\beta(u_2) = -|\log|w_3^2|| = -|\log|\frac{w_3}{\sigma_c(w_3)}|| = -\frac{1}{4}L^*(\chi_2, 0)$. Thus, the image of $e_3(z_3'' \otimes z_3 \otimes z_3')$ in $\zeta(\mathbb{R}[G])$ is $-e_3L_S^*(\chi_2, 0) = -e_3L_S^*(\chi_2, 0)^{\#}$. This shows that the image of $e_2(z_2'' \otimes z_2 \otimes z_2')$ in $\zeta(\mathbb{R}[G])$ is $-e_2L_S^*(\chi_2, 0)^{\#}$.

3.4.6 Equivariant conjecture for $K/\mathbb{Q}$

Theorem 4. *Let $K/\mathbb{Q}$ be the number field extension as above. Then the equivariant Tamagawa number conjecture over number fields holds in the special case $K/\mathbb{Q}$.*

Proof. By the computations in the previous sections it follows that the image in $\zeta(\mathbb{R}[G])^{\times}$ of the generator of $\mathrm{Det}^{-1}(P_{\mathbb{R}}) \otimes \mathrm{Det}(A_{\mathbb{R}} \oplus N_{\mathbb{R}}) \otimes \mathrm{Det}^{-1}(B_{\mathbb{R}})$, obtained by the $\mathbb{Z}[G]$-bases for A, B, P and N is

$$-e_0L_S^*(\chi_0, 0)^{\#} - \zeta_3^2 e_1 L_S^*(\chi_1, 0)^{\#} - e_2L_S^*(\chi_2, 0)^{\#} = (-1, -\zeta_3^2, -1)L_S^*(0)^{\#}.$$

But this is in $\delta^{-1}((A_S, B_S; \psi_S))$. Therefore,

$$\begin{aligned}T\Omega(K/\mathbb{Q}) &= (A_S, B_S; \psi_S) - \delta(L_S^*(0)^{\#}) \\ &= \delta((-1, -\zeta_3^2, -1)L_S^*(0)^{\#}/L_S^*(0)^{\#}) \\ &= \delta((-1, -\zeta_3^2, -1)) \\ &= 0,\end{aligned}$$

where the last equality follows from the fact that the element $(-1, -\zeta_3^2, -1)$ is the image of $-h \in K_1(\mathbb{Z}[G])$. This completes the proof of the Theorem. $\square$

Remark. One could possibly extend the same idea for verifying the conjecture for an infinite family of A_4-extensions. The main idea here would be similar to that of Chinburg ([12, 13]), that is, to look for family of extensions with "subcongruent" unit groups and Ext classes.

Chapter 4

Elliptic Curves

4.1 The setup

Let $K/\mathbb{Q}$ be a finite Galois extension with Galois group G. Let E be an elliptic curve defined over $\mathbb{Q}$. For any field extension L of $\mathbb{Q}$, we let $E_L := E \times_{\mathrm{Spec}(\mathbb{Q})} \mathrm{Spec}(L)$. Our interest is in the motive $M = h^1(E_K)(1)$. Note that the Galois group G acts on M and hence $A := \mathbb{Q}[G]$ acts on the realizations of M.

As noted in Chapter 2, the motivic L-function associated to M is a tuple of twisted Hasse-Weil L-functions. To be precise, for a Dirichlet character χ let $L(E \otimes \chi, s)$ denote the twisted Hasse-Weil L-function. Then we have

$$L(M, s) = (L(E \otimes \chi, s+1))_{\chi \in \widehat{G}} .$$

We shall denote by $L^*(1)$ the leading coefficient

$$L^*(M, 0) = (L^*(E \otimes \chi, 1))_{\chi \in \widehat{G}} .$$

Our aim is to relate this L-value to the arithmetic values arising from a complex as in the conjecture. To compute the arithmetic side, we make certain assumptions that simplify the calculations.

Assumption 1. *$K/\mathbb{Q}$ is totally real and is tamely ramified.*

Assumption 2. *Ш(K) is trivial.*

Assumption 3. *$E(K)$ is trivial.*

Note that $K/\mathbb{Q}$ is tamely ramified if and only if the ring $\mathcal{O}_K$ is a locally free $\mathbb{Z}[G]$-module of rank 1 (cf. [11]). In the example we consider, one has $Cl(\mathbb{Z}[G]) = 1$ and therefore it follows that $\mathcal{O}_K$ is isomorphic to $\mathbb{Z}[G]$ as a $\mathbb{Z}[G]$-module. In the following, we assume in addition to the first assumption that there exists $\alpha_0 \in \mathcal{O}_K$ such that $\mathcal{O}_K \simeq \mathbb{Z}[G]\alpha_0$.

4.2 The arithmetic values

4.2.1 Period isomorphism

The period map α_M is an isomorphism between the Betti and deRham realizations. Therefore two of the terms in the definition of $\Xi(M)$ are trivial. Further, one has $H^1(\mathbb{Q}, M) \simeq E(K) \otimes \mathbb{Q}$, and $H^1_f(\mathbb{Q}, M^*(1))^* \simeq E(K)^* \otimes \mathbb{Q}$, which are both trivial by the second assumption above. Therefore we have

$$\Xi(M) = \boxtimes_{v \in S_\infty} [H_v(M)^{G_v}] \boxtimes [H_{dR}/F^0]. \tag{4.1}$$

Let $\mathcal{E}$ be a Néron model for E over $\mathbb{Z}$. Let ω be a generator of $H^0(\mathcal{E}, \Omega^1_{\mathcal{E}})$. Then the image of the map

$$\begin{array}{rcl} H_1(E(\mathbb{C}), \mathbb{Z}) & \to & \mathbb{C} \\ \gamma & \mapsto & \displaystyle\int_\gamma \omega \end{array}$$

is a $\mathbb{Z}$-lattice in $\mathbb{C}$. We let Ω' to be the least positive real number in this image and we define $\Omega = r\Omega'$ where r is the number of connected components in $E(\mathbb{R})$. This is the real period associated to E (cf. [34]).

Proposition 5. *In the above setting, the image of $\Xi(M)$ in $\mathbb{R}[G]$ under the isomorphism ϑ_∞ is given by*

$$\Omega^{-1}\left(\sum_{g \in G} g(\alpha_0) g^{-1}\right).$$

Proof. Recall that the isomorphism $\vartheta_\infty : \mathbb{R}[G] \simeq \Xi(M) \otimes_{\mathbb{Q}} \mathbb{R}$ is constructed using the period map α_M. Therefore, we shall first write down the Betti and deRham realizations, and the corresponding period map between them.

Let $H_B = \oplus_{\sigma \in \mathrm{Hom}(K,\mathbb{C})} H^1(\sigma E_K(\mathbb{C}), 2\pi i \mathbb{Q})$. We shall identify each summand on the right hand side with the dual homology via the isomorphism

$$H^1(\sigma E_K(\mathbb{C}), 2\pi i\mathbb{Z}) \simeq \mathrm{Hom}(H_1(\sigma E_K(\mathbb{C}), \mathbb{Z}), 2\pi i\mathbb{Z}).$$

Therefore, we have

$$H_B \simeq \mathrm{Hom}\left(\oplus_\sigma H_1(\sigma E_K(\mathbb{C}), \mathbb{Z}), (2\pi i)\mathbb{Z}\right).$$

We shall denote by H_B^+, the fixed submodule of H_B under the action of complex conjugation. Note that H_B^+ is the first term in (4.1).

Let γ_1 and γ_2 be $\mathbb{Z}$-generators of $H_1(E_K(\mathbb{C}), \mathbb{Z})$ such that γ_1 is real, that is fixed under the action of the complex conjugation. Then, $\sigma\gamma_1$ and $\sigma\gamma_2$ are generators for $H_1(\sigma E_K(\mathbb{C}), \mathbb{Z})$. Since K is totally real, it follows that $\sigma\gamma_1$ is real. Now define

$$\tilde{\gamma_1} : \oplus_\sigma H_1(\sigma E_K(\mathbb{C}), \mathbb{Z}) \to (2\pi i)\mathbb{Z}$$

by setting

$$\tilde{\gamma_1}(\sigma\gamma_j) = \begin{cases} 2\pi i & \text{if } \sigma\text{=id, } j = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Note that every embedding of K in $\mathbb{C}$ corresponds to a real infinite place. Therefore, the action of the complex conjugation on these embeddings is trivial. Thus, it follows that $\tilde{\gamma_1}$ generates H_B^+ as a $\mathbb{Q}[G]$-module after identifying via the above isomorphism.

Now, we consider the deRham realization. Note that by Serre duality one has

$$H_{dR}/F^0 = H^1(E_K, \mathcal{O}_{E_K}) \simeq H^0(E_K, \Omega^1_{E_K})^*$$

where $\Omega^1_{E_K}$ is the sheaf of differentials, and $*$ denotes the dual.

Lemma 4. *Let $\mathcal{E}$ and $\mathcal{E}_{\mathcal{O}_K}$ be the Néron models for E over $\mathbb{Z}$ and for E_K over $\mathcal{O}_K$, respectively. Suppose that the conductor of E and the discriminant of K are relatively prime to each other. Then*

$$\mathcal{E}_{\mathcal{O}_K} \simeq \mathcal{E} \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathcal{O}_K).$$

Proof. Note that the primes of bad reduction of E and the primes that ramify in $K/\mathbb{Q}$ do not intersect. Therefore, $\mathcal{E} \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathcal{O}_K)$ is an abelian scheme over $\mathrm{Spec}(\mathcal{O}_K)$. The Lemma now follows from Corollary 1.4 in [2].

□

From the above Lemma we have

$$H^0(E_K, \Omega^1_{E_K}) \simeq H^0(\mathcal{E}_{\mathcal{O}_K}, \Omega^1_{\mathcal{E}_{\mathcal{O}_K}}) \otimes_{\mathcal{O}_K} K \simeq \left(H^0(\mathcal{E}, \Omega^1_{\mathcal{E}}) \otimes_{\mathbb{Z}} \mathcal{O}_K\right) \otimes_{\mathcal{O}_K} K.$$

Let ω_0 be a generator for $H^0(\mathcal{E}, \Omega^1_{\mathcal{E}})$. Then, $((\omega_0 \otimes \alpha_0) \otimes 1)^*$ is a $\mathbb{Q}[G]$-generator for $H_{dR}/\operatorname{Fil}^0$. We shall choose ω_0 such that the period $\int_{\gamma_1} \omega_0$ is precisely Ω.

After identifying the Betti and deRham realizations as above, the period map

$$\alpha_M : H^+_{B,\mathbb{R}} \to H_{dR,\mathbb{R}}/\operatorname{Fil}^0$$

is given by

$$\alpha_M(\gamma \otimes \alpha) : \omega \mapsto \left(\int_\gamma \omega\right)^{-1} \left(\sum_{g \in G} g(\alpha_0) g^{-1}\right).$$

Since $\mathbb{Q}[G]$ is semisimple the determinant functor is given by the map Det defined in Chapter 1. The $\mathbb{Q}[G]$-generators $\tilde{\gamma_1}$ and $((\omega_0 \otimes \alpha_0) \otimes 1)^*$ for H^+_B and $H_{dR}/\operatorname{Fil}^0$ respectively, allows us to identify $\Xi(M)$ inside $\Xi(M)_{\mathbb{R}} \simeq \mathbb{R}[G]$. To be precise, the image of $(\tilde{\gamma_1})^{-1} \otimes ((\omega_0 \otimes \alpha_0) \otimes 1) \cdot \mathbb{Q}[G] = \Xi(M)$ in $\mathbb{R}[G] \simeq \Xi(M)_{\mathbb{R}}$ is $\Omega^{-1}\left(\sum_{g \in G} g(\alpha_0) g^{-1}\right) \cdot \mathbb{Q}[G]$. This completes the proof of the Proposition.

□

4.2.2 The isomorphism ϑ_l

Let $T_l(E) := \varprojlim_n E(\bar{\mathbb{Q}})/l^n$ and let $V_l(E) := T_l(E) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$. Then $V_l(E)$ is the l-adic realization of $M = h^1(E_K)(1)$. Further, $T_l(E)$ is a Galois stable $\mathbb{Z}_l[G]$-lattice sitting inside $V_l(E)$. The algebraic side of Conjecture 2 (4) involves $R\Gamma_c(\mathcal{O}_K[\frac{1}{S}], T_l(E))$, a perfect $\mathbb{Z}_l[G]$-complex, which upon tensoring with $\mathbb{Q}_l$ is quasi-isomorphic to $R\Gamma_c(\mathcal{O}_K[\frac{1}{S}], V_l(E))$.

To compute $[R\Gamma_c(\mathcal{O}_K[\frac{1}{S}], T_l(E))]$ we consider the distinguished triangle

$$R\Gamma_c(\mathcal{O}_K[\frac{1}{S}], T_l(E)) \to R\Gamma_f(K, T_l(E)) \to \oplus_{v \in S} R\Gamma_f(K_v, T_l(E)),$$

where $R\Gamma_f$ is the *finite* component (cf. [8]) of the usual complex $R\Gamma$ of cochains. Following the definitions from [7] and [8], we have that for $v \nmid l\infty$, $R\Gamma_f(K_v, T_l)$ is quasi-isomorphic to

$$T_l(E)^{I_v} \xrightarrow{1-\mathrm{Fr}_v} T_l(E)^{I_v}$$

where the modules are placed in degree 0 and 1. Further, for $v|\infty$, the complex $R\Gamma_f(K_v, T_l)$ is defined to be the complex $R\Gamma(K_v, T_l)$. Finally, for $v|l$, we note that

$$H^1_f(K_v, V_l(E)) \simeq (\varprojlim_n E(K_v)/l^n) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \simeq t_E \otimes_{\mathbb{Q}} \mathbb{Q}_l$$

where t_E is the tangent space of E_K and the latter isomorphism is given by the exponential map. The $Z_l[G]$-submodule $\varprojlim_n E(K_v)/l^n$ can have torsion points, and therefore is not a good choice for $H^1_f(K_v, T_l(E))$. Let $\mathcal{E}_{\mathcal{O}_K}$ be a Néron model for E_K. Then there is a free 1-dimensional $\mathbb{Z}[G]$-lattice $H^0(\mathcal{E}_{\mathcal{O}_K}, \Omega^1_{\mathcal{E}_{\mathcal{O}_K}})$ sitting inside $t_E \simeq H^0(E_K, \Omega^1_{E_K})$. We set

$$H^1_f(K_v, T_l(E)) = H^0(\mathcal{E}_{\mathcal{O}_K}, \Omega^1_{\mathcal{E}_{\mathcal{O}_K}}) \otimes_{\mathbb{Z}} \mathbb{Z}_l.$$

This gives a projective $\mathbb{Z}_l[G]$-lattice sitting insider $H^1_f(K_v, V_l(E))$.

Lemma 5. *In the above setting one has*

1. *For $v|\infty$, the complex $R\Gamma_f(K_v, T_l(E))$ is quasi-isomorphic to $H_1(\sigma E(\mathbb{C}), \mathbb{Z}) \otimes \mathbb{Z}_l$ (in degree 0) if $K_v \simeq \mathbb{C}$ and is quasi-isomorphic to $H_1(\sigma E(\mathbb{C}), \mathbb{Z})^+ \otimes \mathbb{Z}_l$ (in degree 0) if $K_v \simeq \mathbb{R}$. Here σ is the embedding corresponding to the infinite prime v.*

2. *For $v \nmid l\infty$ $\#H^1_f(K_v, T_l(E))$ is finite and is given by the local Tamagawa numbers.*

3. *For $v|l$ one has*

$$\begin{aligned} H^1_f(K_v, T_l(E)) &\simeq H^0(\mathcal{E}_{\mathcal{O}_K}, \Omega^1_{\mathcal{E}_{\mathcal{O}_K}}) \otimes_{\mathbb{Z}} \mathbb{Z}_l \\ &\subset H^0(E_K, \Omega^1_{E_K}) \otimes_{\mathbb{Q}} \mathbb{Q}_l \simeq H^1_f(K_v, V_l) \end{aligned}$$

Further, for all $v \in S$ there is an inclusion

$$R\Gamma_f(K_v, T_l(E)) \to R\Gamma_f(K_v, V_l(E)),$$

which becomes a quasi-isomorphism upon tensoring with $\mathbb{Q}_l$.

The finite cohomology in the global case is given by the following Proposition.

Lemma 6. *Assume that $\text{Ш}(E_K)$ is finite and that $l \neq 2, 3$. Then,*

$$H^i_f(K, T_l(E)) \simeq \begin{cases} 0 & \text{if } i = 0 \\ E(K) \otimes_{\mathbb{Z}} \mathbb{Z}_l & \text{if } i = 1 \\ \mathrm{Hom}_{\mathbb{Z}}(E(K)_{l^\infty}, \mathbb{Q}_l/\mathbb{Z}_l) & \text{if } i = 3 \end{cases}$$

and

$$0 \to \text{Ш}'_{l^\infty} \to H^2_f(K, T_l(E)) \to \mathrm{Hom}_{\mathbb{Z}}(E(K), \mathbb{Z}_l) \to 0$$

is a short exact sequence. Here $\text{Ш}'$ is a group that differs from the Shafarevich-Tate group by the torsion of $\varprojlim_n E(K_v)/l^n$.

See [7] or [22] for proofs of the above Lemmas.

Remarks.

1. The quasi-isomorphism

$$\oplus_{v \in S_\infty} R\Gamma_f(K_v, T_l(E)) \otimes \mathbb{Q}_l \to \oplus_{v \in S_\infty} R\Gamma_f(K_v, V_l(E))$$

induces as isomorphism

$$\oplus_{\sigma: K \to \mathbb{C}} H_1(\sigma E(\mathbb{C}), \mathbb{Z})^+ \otimes \mathbb{Q}_l \simeq \mathrm{Det}_{\mathbb{Q}_l[G]} \oplus_{v \in S_\infty} R\Gamma_f(K_v, V_l(E)),$$

which is precisely the comparison isomorphism between the étale and singular cohomologies.

2. If $E(K)$ and $\text{Ш}(E/K)_{l^\infty}$ are trivial then $R\Gamma_f(K, T_l)$ is quasi-isomorphic to the zero complex and hence

$$[R\Gamma_c(\mathcal{O}_K[\frac{1}{S}], T_l(E))] = [\text{Ш}'_{l^\infty}] \boxtimes [\oplus_{v \in S} R\Gamma_f(K_v, T_l(E))].$$

Note that the contribution of $[\text{Ш}'_{l^\infty}]$ is just the torsion of $\varprojlim_n E(K_v)/l^n$.

With the above Lemmas one quickly deduces the following.

Proposition 6. *In the above setting suppose that $\text{Ш}(K)$ and $E(K)$ are trivial. The image of $[R\Gamma_c(\mathcal{O}_K[\frac{1}{S}], T_l(E))]$ in $[R\Gamma_c(\mathcal{O}_K[\frac{1}{S}], V_l(E))]$ is the $\mathbb{Z}_l[G]$-line generated by $(\omega_0 \otimes \alpha_0)$.*

4.3 Special values of abelian twists

4.3.1 Modular symbols

Let $\mathcal{H} = \{z \in \mathbb{C} | \mathrm{Im}(z) > 0\}$ denote the upper half plane and let $\mathcal{H}^* = \mathcal{H} \cup \mathbb{Q} \cup \{\infty\}$ denote the extended upper half plane. The group $SL_2(\mathbb{Z})$ acts discretely and discontinuously on $\mathcal{H}^*$. For any congruence subgroup Γ of $SL_2(\mathbb{Z})$ let X_Γ be the modular surface $\Gamma \backslash \mathcal{H}^*$. Further, let $S_2(\Gamma)$ be the space of all cusp forms of weight 2 for Γ. There is a bilinear pairing

$$\begin{array}{rcl} S_2(\Gamma) \times H_1(X_\Gamma, \mathbb{C}) & \to & \mathbb{C} \\ (f, \gamma) & \to & 2\pi i \displaystyle\int_\gamma f(z)dz \end{array}$$

that induces an isomorphism $S_2(\Gamma) \simeq H_1(X_\Gamma, \mathbb{C})^{+\wedge}$ of complex vector spaces.

For cusps $\alpha, \beta \in \mathbb{Q} \cup \{\infty\}$ consider a smooth path in $\mathcal{H}^*$ from α to β. For a congruence subgroup Γ of $SL_2(\mathbb{Z})$ let $\{\alpha, \beta\}_\Gamma$ denote the image of the path in X_Γ. Note that if α and β are Γ-equivalent, then $\{\alpha, \beta\}_\Gamma$ is a closed path in X_Γ and hence defines an element of $H_1(X_\Gamma, \mathbb{Z})$. The Manin-Drinfeld Theorem says that for any cusps $\alpha, \beta \in \mathbb{Q} \cup \{\infty\}$ one has $\{\alpha, \beta\}_\Gamma \in H_1(X_\Gamma, \mathbb{Q})$. Note that this element is independent of the choice of the path from α and β. Therefore, we can (and will) identify $\{\alpha, \beta\}_\Gamma$ as an element of $H_1(X_\Gamma, \mathbb{Q})$. We shall also drop the subscript Γ whenever the notation is unambiguous. For a cusp form $f \in S_2(\Gamma)$ let

$$\langle \{\alpha, \beta\}_\Gamma, f \rangle := 2\pi i \int_\alpha^\beta f(z)dz$$

Again, note that this definition is independent of the choice of the path from α to β.

For $\gamma \in SL_2(\mathbb{Z})$ let $(\gamma) = \{\gamma(0), \gamma(\infty)\}$. Let $C(\Gamma)$ be a $\mathbb{Q}$-vector space the symbols (γ) as basis, where γ runs over a set of representatives for the cosets of $SL_2(\mathbb{Z})/\Gamma$. Let R be the left ideal of $\mathbb{Z}[SL_2(\mathbb{Z})]$ generated by $I + S$ and $I + (TS) + (TS)^2$, where

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Let $B(\Gamma) = C(\Gamma)R$. Let $H_0(\Gamma)$ denote the free abelian group on the cusps of Γ. Let $Z(\Gamma)$

denote the kernel of the map $C(\Gamma) \to H_0(\Gamma)$ defined by

$$(\gamma) \to [\gamma(\infty)] - [\gamma(0)]$$

where $[\alpha]$ denotes the Γ-orbit of the cusp α. Then, $B(\Gamma) \subset Z(\Gamma)$. Let $H(\Gamma) = Z(\Gamma)/B(\Gamma)$.

Proposition 7. *As $\mathbb{Q}$-vector spaces $H(\Gamma)$ and $H_1(X_\Gamma, \mathbb{Q})$ are isomorphic, and the isomorphism is given by*

$$(\gamma) \mapsto \{\gamma(0), \gamma(\infty)\}_\Gamma.$$

Proof. See [26]. □

4.3.2 Hecke operators

For a prime p let $\mathcal{T}_p$ denote the Hecke operator. Thus, for a cusp for $f \in S_2(\Gamma)$, the Hecke operator acts as

$$f|\mathcal{T}_p = f \left| \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} + \sum_{k=0}^{p-1} f \right| \begin{pmatrix} 1 & k \\ 0 & p \end{pmatrix}.$$

Now, define the action of these Hecke operators on $H_1(X_\Gamma, \mathbb{Q})$ by

$$\mathcal{T}_p\{\alpha, \beta\} = \{p\alpha, p\beta\} + \sum_{k=0}^{p-1} \left\{ \frac{\alpha + k}{p}, \frac{\beta + k}{p} \right\}.$$

Then one has

$$\langle \{\alpha, \beta\}, f|\mathcal{T}_p \rangle = \langle \mathcal{T}_p\{\alpha, \beta\}, f \rangle.$$

4.3.3 L-values

Henceforth, we let $\Gamma = \Gamma_0(N)$ and $f \in S_2(\Gamma)$. We first note that the Mellin transform for the L-function attached to a new form f is given by

$$L(f, s) = (2\pi)^s \Gamma(s)^{-1} \int_0^{i\infty} (-iz)^s f(z) \frac{dz}{z}.$$

Substituting $s = 1$ in the above we get

$$L(f, 1) = -2\pi i \int_0^{i\infty} f(z) dz = -\langle \{0, \infty\}, f \rangle.$$

Note that a similar identity holds for twisted forms $f \otimes \chi$.

Proposition 8. *Let f be a new form of level N and let χ be a Dirichlet character of conductor l such that $(l, N) = 1$. Then we have*

$$L(f \otimes \chi, 1) = \frac{g(\chi)}{l} \sum_{a=1}^{l} \bar{\chi}(a) \left\langle \{0, a/l\}, f \right\rangle,$$

where $g(\chi) = \sum_{n=1}^{l} \chi(n)\zeta_l^n$ is the Gauss sum.

Proof. Note that, by the definition of the Gauss sum one has

$$\bar{\chi}(-a)g(\chi) = \sum_{n=1}^{l} \chi(-a^{-1}n)\zeta_l^n = \sum_{n=1}^{l} \chi(n)\zeta_l^{-an}.$$

Therefore we get

$$\chi(n) = \frac{g(\chi)}{l} \sum_{a=1}^{l} \bar{\chi}(-a)\zeta_l^{an}.$$

Now, if $f(z) = \sum_{n \geq 1} a_n e^{2\pi i n z}$ is the q-expansion of f then

$$\begin{aligned}
(f \otimes \chi)(z) &= \sum_{n \geq 1} \chi(n) a_n e^{2\pi i n z} \\
&= \sum_{n \geq 1} \frac{g(\chi)}{l} \sum_{a=1}^{l} \bar{\chi}(-a) a_n e^{2\pi i \left(\frac{lz+a}{l}\right)} \\
&= \left(\frac{g(\chi)}{l} \sum_{a=1}^{l} \bar{\chi}(-a) f|_{\gamma_a} \right)(z),
\end{aligned}$$

where $\gamma_a = \begin{pmatrix} l & a \\ 0 & l \end{pmatrix}$.

Now since $L(f \otimes \chi, 1) = -2\pi i \int_0^\infty (f \otimes \chi)(z)dz = -\langle \{0, \infty\}, f \otimes \chi \rangle$, we get

$$\begin{aligned} L(f \otimes \chi, 1) &= -\langle \{0, \infty\}, f \otimes \chi \rangle \\ &= -\frac{g(\chi)}{l} \sum_{a=1}^{l} \bar{\chi}(a) \langle \{0, \infty\}, f|_{\gamma_a} \rangle \\ &= -\frac{g(\chi)}{l} \sum_{a=1}^{l} \bar{\chi}(a) \langle \{\gamma_a(0), \gamma_a(\infty)\}, f \rangle \\ &= -\frac{g(\chi)}{l} \sum_{a=1}^{l} \bar{\chi}(a) \langle \{a/l, \infty\}, f \rangle \\ &= \frac{g(\chi)}{l} \sum_{a=1}^{l} \bar{\chi}(a) \langle \{0, a/l\}, f \rangle \,. \end{aligned}$$

This completes the proof of the Proposition. □

Recall that we have an isomorphism $S_2(\Gamma) \simeq H_1(X_\Gamma, \mathbb{C})^{+\wedge}$. Thus, a rational new form $f \in S_2(\Gamma)$ corresponds to an element $\{\alpha, \beta\} \in H_1(X_\Gamma, \mathbb{Q})^+$ via this isomorphism. Note that this element is unique up to sign if we further restrict it to be an element of $H_1(X_\Gamma, \mathbb{Z})^+$. We shall denote this 1-cycle by γ_f. Since the pairing $\langle , \rangle$ is compatible with the action of the Hecke operators, it follows that γ_f is a common eigen vector for all the Hecke operators with eigenvalues same as that of the new form. Thus, by looking at the Hecke action on $H_1(X_\Gamma, \mathbb{Z})^+$ we can compute γ_f. Let V be the $\mathbb{Q}$-vector space generated by γ_f. Note that we can construct the complementary subspace V' on which the pairing $\langle \cdot, f \rangle$ is trivial. Then, for any $\gamma \in H_1(X_\Gamma, \mathbb{Q})^+$ we have

$$\int_\gamma f = \int_{\gamma|_V} f$$

where $\gamma|_V$ is the projection of γ onto the subspace V. In fact, $\gamma|_V$ is a rational multiple of γ_f. So, if we let $\Omega_f := \int_{\gamma_f} f$ then we see that $\int_\gamma f$ is a rational multiple of Ω_f. The following well-known Proposition gives the equality between the periods of an elliptic curve and the corresponding modular form.

Proposition 9. *Let E be a strong Weil curve defined over $\mathbb{Q}$ and let f be the normalized rational new form corresponding to E. Then, the real period associated to E equal $c_E d\Omega_f$, where d is the number of components in the real locus of E (that is, $d = 2$ if the corresponding lattice is rectangular, $d = 1$ otherwise) and c_E is the Manin constant (known to be trivial for $X_0(p)$).*

Note that, $\{0, a/l\} + \{0, (l-a)/l\} \in H_1(X_\Gamma, \mathbb{Z})^+$ and $\{0, \infty\} \in H_1(X_\Gamma, \mathbb{Z})^+$. Hence we can compute the L-values of a rational new form and its twists using the above computation. Implementation of such a computation is studied in detail by Cremona (see [16]).

If τ is a nonabelian irreducible representation of G, then a formula such as (4.1) does not exist. However, by a Theorem of Brauer one can write τ as a linear combination of representations induced by abelian representations of the subgroups of G. Therefore one has

$$L(f \otimes \tau, s) \quad = \quad \prod_i L(\widehat{f}_i \otimes \tau_i, s) \tag{4.2}$$

where $\widehat{f}_i$ is the base change of f to the fixed field of a subgroup G_i of G, and τ_i is an abelian representation of G_i. Thus if one could get a formula for $L(\widehat{f}_i \otimes \tau_i, s)$ analogous to (4.1), then we can compute all the special values of the twisted L-functions. However, the theory of modular symbols is lot more complicated even for a quadratic field. For example, one can write down an infinite set of generators for $H_2(X, \mathbb{Q})$ (where X is the modular surface), but the relations amongst these generators are completely unknown (cf. [29]).

The special values of twisted L-functions associated to a base change form $\widehat{f}$ of f are related to the periods of $\widehat{f}$. The equation (4.2) therefore relates the special value of a twist of f and the periods of base change forms. Note that the arithmetic side of the equivariant conjecture involves only the periods associated to f. Thus, the equivariant conjecture implies period relations between the periods of f and periods of the base change forms. These relations are precisely the ones conjectured by Doi, Hida and Ishii in [20].

4.4 Special values of nonabelian twists

Let τ be an (irreducible) self-dual representation of G and let $N(E, \tau)$ be the conductor of $E \otimes \tau$. We suppose that the L-function $L(E \otimes \tau, s)$ has a meromorphic continuation to the whole s-plane and that it satisfies a functional equation

$$\widehat{L}(E \otimes \tau, s) = \pm \widehat{L}(E \otimes \tau, 2-s) \tag{4.3}$$

where

$$\widehat{L}(E \otimes \tau, s) = A^s \gamma(s) L(E \otimes \tau, s)$$

for some constant $A = A(E, \tau)$ and a Γ-factor

$$\gamma(s) = \Gamma\left(\frac{s+\lambda_1}{2}\right)\Gamma\left(\frac{s+\lambda_2}{2}\right)\cdots Gamma\left(\frac{s+\lambda_r}{2}\right).$$

These assumptions are known to hold in many cases. For instance, if τ is a Dirichlet character, then the assumptions are know to be true due to Shimura. Bouganis and Dokchitser have shown in [4] that the above assumption is true for the nonabelian representation of a false Tate extension. In these cases, one has $A = \frac{\sqrt{N(E,\tau)}}{\pi^d}$, where d is the dimension of τ. Also, one has

$$\gamma(s) = \Gamma\left(\frac{s}{2}\right)^d \Gamma\left(\frac{s+1}{2}\right)^d.$$

Now, let $\phi(s)$ be the inverse Mellin transform of $\gamma(s)$, that is,

$$\gamma(s) = \int_0^\infty \phi(t)t^s \frac{dt}{t}.$$

Let

$$G_s(t) = t^{-s}\int_t^\infty \phi(x)x^s \frac{dx}{x}$$

be the incomplete Mellin transform of $\phi(t)$. Then one has

Proposition 10.

$$\widehat{L}(E\otimes\tau, s) = \sum_{n=1}^{\infty} a_n G_s\left(\frac{n}{A}\right) \pm \sum_{n=1}^{\infty} a_n G_{2-s}\left(\frac{n}{A}\right). \tag{4.4}$$

Proof. See [19] or [36]. □

For fixed s, the series (4.4) converges exponentially with t and therefore we can use this series to get numerical approximations of the value $L(E\otimes\tau, 1)$. The rate of convergence of the series depends on the conductor $N(E\otimes\tau)$. We roughly need to sum $\sqrt{N(E\otimes\tau)}$ terms in the series to obtain an approximation. Note that if the bad primes of E and τ do not intersect, then the conductor of $E\otimes\tau$ is $N(E,\tau) = N(E)^d N(\tau)^2$. Therefore, obtaining numerical approximations to the value $L(E\otimes\tau, 1)$ is computationally infeasible for large field extensions.

In [19] Dokchitser has explained how to compute $G_s(t)$ efficiently and this has been implemented in [15]. Our numerical approximations use this particular implementation.

Remark. The Proposition 10 is in fact a special case of a more general result that holds for any L-series having a meromorphic continuation and satisfying a functional equation of type (4.3).

4.5 An example

Let $p(x) = x^3 - 4x + 1$, and let K be the splitting field of $p(x)$ over $\mathbb{Q}$. The discriminant of $p(x)$ is 229, and hence K is a totally real S_3-extension. The unique quadratic subfield of K is $\mathbb{Q}[\sqrt{229}]$.

Let r and s be elements of $G := S_3$ of order 2 and 3 respectively. There are three irreducible characters of G, two of which are abelian and the third has dimension 2. The character table is shown below.

	1	r	s
1	1	1	1
χ	1	-1	1
ψ	2	0	-1

Here χ is the nontrivial abelian character and ψ is the character of the irreducbile nonabelian representation ρ of G given by

$$r \mapsto \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad s \mapsto \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$

In fact, ψ is induced by the trivial character on $C_3 = \{1, s, s^2\} \subset S_3$, that is, $\psi = \mathrm{Ind}_{C_3}^{S_3} 1_{C_3}$. So for any new form $f \in S_2(\Gamma_0(N))$ one has

$$L(f \otimes \psi, s) = L(\widehat{f}, s)$$

where $\widehat{f}$ is the base change of f to $\mathbb{Q}[\sqrt{229}]$.

Let E be the elliptic curve defined by $y^2 + y = x^3 - 8x - 9$. This is the curve $307A1$ in the sense of [16]. Note that this is a strong Weil curve and hence all the above computations are applicable to this curve. The conductor of the curve is 307, and as noted in [16] the group $E(\mathbb{Q})$ is trivial. We note that the primes 37 and 53 split completely in K. It is easy to check that $E(\mathbb{F}_{37}) = 35$ and $E(\mathbb{F}_{53}) = 64$, and therefore it follows that $E(K)_{\mathrm{tors}}$ is trivial (cf.

[34]). As we will see in the next section, the L-functions $L(E\otimes\tau,s)$ do not vanish at $s=1$ for any $\tau\in\widehat{G}$ and therefore one can conclude that $E(K)$ is trivial. Thus, $M=h^1(E_K)(1)$ satisfies the first two assumptions made in Section 4.1.

4.5.1 Analytic values

The special value of the L function attached to M is $L^*(M,0):=(L^*(E\otimes\phi,1))_{\phi\in\widehat{G}}$. In fact, we can view this as an element of $\zeta(\mathbb{R}[G])^\times$:

$$L^*(M,0)=\sum_{\phi\in\widehat{G}}L^*(E\otimes\phi,1)e_\phi$$

where $e_\phi=\frac{1}{|G|}\sum_{g\in G}\phi(g)g^{-1}$ is the idempotent corresponding to the character $\phi\in\widehat{G}$.

In our case, we have $G=S_3$, and $\widehat{G}=\{1,\chi,\psi\}$. The L-values $L(E\otimes\phi,1)$ are nonvanishing for all the characters $\phi\in\widehat{G}$ and thus, $L^*(E\otimes\phi,1)=L(E\otimes\phi,1)$. Since the $E(K)$ has no torsion and the L-function does not vanish at $s=1$, it follows that $E(K)$ is trivial. Thus, E and K satisfy the necessary hypotheses stated in Section 4.1. For $\phi=1,\chi$ the L-value can be computed using modular symbols as described in Section 4.3.1. The computations yield:

$$L(E,1)=\Omega_E, L(E\otimes\chi,1)=-\Omega_E/\sqrt{229}$$

where Ω_E is the real period attached to E.

We apply methods described in the Section 4.4 to compute $L(E\otimes\psi,1)$. The conductor of $E\otimes\psi$ is $N(E)^2\cdot N(\psi)^2$, where $N(E)=307$ and $N(\psi)=229$ are the condutors of E and ψ respectively. The coefficients of the sereis $L(E\times\psi,s)$ can be computed by looking at the local factors:

$$L_p(E\otimes\psi,T)=\begin{cases}1+T+T^2 & \text{if } p=307\\ 1-a_pT+pT^2 & \text{if } p=229\\ (1-a_T+pT^2)^2 & \text{if } p\neq 307,229 \text{ and } f(p)=1\\ 1+(2p-a_p^2)T^2+p^2T^4 & \text{if } p\neq 307,229 \text{ and } f(p)=2\\ 1+a_pT+(a_p^2-p)T^2+a_ppT^3+p^2T^4 & \text{if } p\neq 307,229 \text{ and } f(p)=3\end{cases}$$

The computations yield $L(E\otimes\psi,1)\sim 0.1330438452$.

4.5.2 Arithmetic values

We first consider the period isomorphism. Recall from Section 4.2 that the period isomorphism

$$\alpha_M : H^+_{B,\mathbb{R}} \to H_{dR,\mathbb{R}}/F^0$$

is given by

$$\alpha_M(\gamma \otimes \alpha) : \omega \mapsto \left(\int_\gamma \omega\right)^{-1} \left(\sum_{g \in G} g(\alpha) g^{-1}\right). \tag{4.5}$$

Lemma 7. *Consider the complex* $\Phi : \mathbb{R}[G] \xrightarrow{\theta} \mathbb{R}[G]$ *of* $\mathbb{R}[G]$*-modules defined in degree 0 and 1,* θ *being an isomorphism. Then the image of the trivial generator under the isomorphism* $\mathrm{Det}(\Phi) \simeq \zeta(\mathbb{R}[G])$ *is*

$$\sum_{\eta \in \widehat{G}} e_\eta(\det \rho_\eta(\theta(1)))$$

where ρ_η *is an irreducible representation with character* η*.*

Proof. Let ρ be an irreducible complex representation of G and η be the corresponding character. Let $e_\eta = \sum_{g \in G} \eta(g) g^{-1}$, the idempotent associated to η. Then $e_\eta \mathbb{C}[G] \simeq M_n(\mathbb{C})$ for some $n \geq 1$. Let

$$e = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ .. & .. & \dots & .. \\ .. & .. & \dots & .. \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

Then $eM_n(\mathbb{C})$ is an n-dimensional vector space over $\mathbb{C}$. Note that $e_\eta \theta(1) = \theta(e_\eta) \in e_\eta \mathbb{C}[G] \simeq M_n(\mathbb{C})$, so let $\rho(\theta(1)) = e_\eta \theta(1) = (\alpha_{ij})$. Then by the construction of the determinant functor for semisimple in Section 2.1.2, we have $e_\eta \operatorname{Det} \Phi$ is the $\mathbb{C}$-line bundle generated by determinant of the map

$$\bar{\theta} : eM_n(\mathbb{C}) \longrightarrow eM_n(\mathbb{C}).$$

The determinant of $\bar{\theta}$ is given by $\det(\alpha_{ij}) = \det(\rho(\theta(1)))$ and therefore $\mathrm{Det}(\Phi)$ equals $\sum_{\eta \in \widehat{G}} e_\eta \det(\rho(\theta(1)))$. $\square$

Note that $(\omega_0 \otimes \alpha_0)$ and $(\gamma_0 \otimes 1)$ are the $\mathbb{Q}[G]$-generators which define $\Xi(M)$ inside

$\Xi(M)\otimes\mathbb{R}$. From the above Lemma and (4.5), it follows that the image of $\Xi(M)$ in $\Xi(M)\otimes\mathbb{R}$ is given by the $\mathbb{Q}[G]$-generator

$$R\Omega := \sum_{\eta\in\widehat{G}} \det(\rho_\eta(\Omega_E^{-1}\sum_{g\in G} g(\alpha_0)g^{-1}))e_\eta.$$

After choosing a particular $\mathbb{Z}[G]$-generator α_0 of $\mathcal{O}_K$ (using PARI), we get the following components:

$$\begin{aligned}
e_1 R\Omega &= \Omega_E^{-1}\cdot \mathrm{tr}_{K/\mathbb{Q}}(\alpha_0) = \Omega_E^{-1} = L(E,1)^{-1} \\
e_\chi R\Omega &= \Omega_E^{-1}\cdot \mathrm{tr}_{K/\mathbb{Q}[\sqrt{229}]}(\alpha_0 - g_1(\alpha_0)) = \Omega_E^{-1}\cdot\sqrt{229} = -L(E\otimes\chi,1)^{-1} \\
e_\psi R\Omega &\sim \Omega_E^{-2}\cdot(\sqrt{229}) \sim (0.7047640380)^2(15.131274594) = 7.516319136 \\
&\sim L(E\otimes\psi,1)^{-1}(1.00000000)
\end{aligned}$$

4.5.3 Equivariant conjecture and consequences

As elements of $\zeta(\mathbb{R}[G])^\times$ we have

$$L^*(M,0)R\Omega \sim e_1 - e_\chi + e_\psi, \tag{4.6}$$

which is an element of $\zeta(\mathbb{Q}[G])^\times$. Thus $L^*(M,0)^{-1}$ and $R\Omega$ generate the same $\mathbb{Q}[G]$-line in $\Xi(M)\otimes\mathbb{R}$. This is in accordance with the rationality part of the equivariant Conjecture. Note that the right hand side of (4.6) is a unit in $\mathbb{Z}_l[G]$ for $l\neq 2,3$. Therefore (4.6) provides a numerical verification for the l-part of Conjecture 2 (4), for $l\neq 2,3$. Our calculations in Section 4.2 excluded the primes 2 and 3, and therefore we cannot conclude anything about 2-part or 3-part of the conjecture. To summarize, we have the following.

Theorem 5. *Let K be the splitting field of $p(x) = x^3-4x+1$, and let E be the elliptic curve defined by $y^2+y = x^3-8x-9$. Then the above calculations provide a numerical verification for the l-part of equivariant Conjecture (with $l\neq 2,3$), for the motive $M = h^1(E_K)(1)$ with coefficients in $\mathbb{Q}[G]$.*

Remarks.

1. For abelian extensions $K/\mathbb{Q}$, the methods of modular symbols give the precise value of L-function in terms of the period, and thus one can possibly prove the equivariant

Conjecture in the abelian case. The above result in fact proves the Conjecture (under the assumption that Ш is trivial) for $h^1(E_{\mathbb{Q}[\sqrt{229}]})(1)$.

2. The assumption on the triviality of $E(K)$ can possibly be removed by carrying out the calculations in Section 4.2 more carefully. Also, by Lemma 5 in [8], one can possibly verify the conjecture for a curve with nontrivial Ш (or nontrivial torsion group) by considering an isogenous curve that has trivial Ш and torsion group.

Appendix A

An Auxiliary Lemma

We shall prove here the following lemma:

Lemma 8. *The map*

$$K_0(\mathbb{Z}[A_4],\mathbb{R}) \to K_0(\mathfrak{M}_{A_4},\mathbb{R}) \times K_0(\mathbb{Z}[A_4]) \times K_0(\mathbb{Z}[A_4^{ab}],\mathbb{R})$$

is not injective, where $\mathfrak{M}_{A_4}$ is a maximal order in $\mathbb{Q}[A_4]$ containing $\mathbb{Z}[A_4]$.

Proof. Following the notations in the proof of Lemma 4 of [9], the above map is injective if and only if the map $D(A_4) \to D(C_3)$ is injective where for a finite group Γ one defines

$$D(\Gamma) := \frac{\mathrm{nr}(U_f(\mathfrak{M}_\Gamma)) \cap \mathrm{im}(\mathrm{nr}_{\mathbb{Q}[\Gamma]})}{\mathrm{nr}(U_f(\mathbb{Z}[\Gamma])) \cap \mathrm{im}(\mathrm{nr}_{\mathbb{Q}[\Gamma]})},$$

with $\mathfrak{M}_\Gamma$ being a maximal order in $\mathbb{Q}[\Gamma]$ containing $\mathbb{Z}[\Gamma]$.

The group $D(C_3)$ is isomorphic to $\mathfrak{M}_{C_3}^\times/\mathbb{Z}[C_3]^\times$ which is of order 2. The numerator of $D(A_4)$ is $\mathbb{Z}^\times \times \mathbb{Z}[\zeta_3]^\times \times \mathbb{Z}^\times$ since $\mathbb{Q}[A_4] \simeq \mathbb{Q} \times \mathbb{Q}[\zeta_3] \times M_3(\mathbb{Q})$. Thus, the numerator of $D(A_4)$ has order 24. The denominator is $\{\mathrm{nr}(u) : u \in \mathbb{Z}[A_4]^\times\}$. If we show that this has order 6, then it follows that $D(A_4)$ has order 4, so the map $D(A_4) \to D(C_3)$ is not injective and hence the lemma follows.

Now consider a unit $u \in \mathbb{Z}[A_4]^\times$. The image of this under the norm reduction map nr lies inside $\mathbb{Z}^\times \times \mathbb{Z}[\zeta_3]^\times \times \mathbb{Z}^\times$. Let $\mathrm{nr}(u) = (a, b, c)$ under this identification. We shall show that $a = c$.

Note that $\mathrm{nr}(u) = (\rho_0(u), \rho_1(u), \det(\rho_2(u)))$, where ρ_0 is the trivial representation, ρ_1 is a nontrivial abelian representation and ρ_2 is the degree three representation of A_4. By replacing u by $-u$ if necessary, we can assume that $a = 1$. Further, we can assume that the image of u under the projection $\mathbb{Z}[A_4]^\times \to \mathbb{Z}[C_3]^\times$ is trivial, by multiplying by a suitable unit $(\pm h^i)$ if necessary. Thus, if $u = \sum_{g\in A_4} a_g \cdot g$ where $a_g \in \mathbb{Z}$, then $\sum_{g\in A_4} a_g = \rho_0(u) = a = 1$. Further, the image of u in $\mathbb{Z}[C_3]^\times$ is trivial implies that $a_e + a_{g_1} + a_{g_2} + a_{g_3} = 1$ and $a_{h^i} + \sum_{j=1}^{3} a_{h^i g_j} = 0$ for $i = 1, 2$.

The representation ρ_2 is given by the following:

$$g_1 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \; g_2 \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \; h \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Thus, $c = \rho_2(u)$ equals the determinant of $(r_{ij})_{1\le i,j\le 3}$ where

$$r_{ij} = 2(a_{h^{-i+j}} + a_{h^{-i+j}g_j}) - (a_{h^{-i+j}} + a_{h^{-i+j}g_1} + a_{h^{-i+j}g_2} + a_{h^{-i+j}g_3}).$$

Therefore, by the above identities, c equals the determinant of

$$\begin{pmatrix} 2(a_e + a_{g_1}) - 1 & 2(a_h + a_{hg_2}) & 2(a_{h^2} + a_{h^2 g_3}) \\ 2(a_{h^2} + a_{h^2 g_1}) & 2(a_e + a_{g_2}) - 1 & 2(a_h + a_{hg_3}) \\ 2(a_h + a_{hg_1}) & 2(a_{h^2} + a_{h^2 g_2}) & 2(a_e + a_{g_3}) - 1 \end{pmatrix}.$$

Looking at the determinant of the above modulo 4, we see that $c \equiv 1 \pmod 4$ and hence $c = a$. Therefore, we have

$$\#\{\mathrm{nr}(u) : u \in \mathbb{Z}[A_4]^\times\} = \#\{(\rho_0(u), \rho_1(u)) : u \in \mathbb{Z}[A_4]^\times\} = \#\mathbb{Z}[C_3]^\times = 6.$$

This completes the proof.

$\square$

Appendix B

A Brief History of Equivariant Conjecture

The following table gives a brief history of the various results and conjectures related to the special values of L-functions that eventually led to the formulation of the equivariant Tamagawa number conjecture.

Year	Mathematician(s)	Result or Conjecture
1838	Dirichlet	Analytic class number formula for quadratic fields
1850	Kummer	Analytic class number formula for cyclotomic fields
1859	Riemann	Riemann hypothesis
1896	Dedekind	Analytic class number formula for number fields
1917	Hecke	Analytic continuation of Dedekind ζ-function
1949	Weil	Weil conjectures
1960	Dwork	Rationality part of Weil conjectures
1963	Ono	Tamagawa number of tori
1965	Birch, Swinnerton-Dyer	Birch and Swinnerton-Dyer (BSD) conjecture
1966	Tate	Generalization of BSD conjecture, Tate's conjecture
1969	Serre	Definition of motivic L-functions
1972	Quillen	Definition of algebraic K-groups
1974	Borel	Relation between $\zeta_K(n)$ and $K_{2n-1}(\mathcal{O}_K)$
1974	Deligne	Proof of Weil conjectures
1976	Coates, Wiles	Rank part of BSD conj. in rank zero, CM case
1978	Bloch	$K_2(E) \sim L(E,2)$ for an elliptic curve E
1979	Deligne	Rationality conjecture for $L(M,0)$ for critical M

Year	Mathematician(s)	Result or Conjecture
1983	Gross, Zagier	Rank part of BSD conj. when $L(E,1)=0$
1985	Beilinson	Rationality conjecture in a general setting
1988	Bloch, Kato	Integrality conjecture for $L(M,s)$
1990	Kolyvagin	Rank part of BSD conj. when $L(E,1)\neq 0$
1991	Fontaine, Perrin-Riou	Reformulation of Bloch-Kato conjecture
1996	Burns, Flach	Equivariant conjecture in the nonabelian setting

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